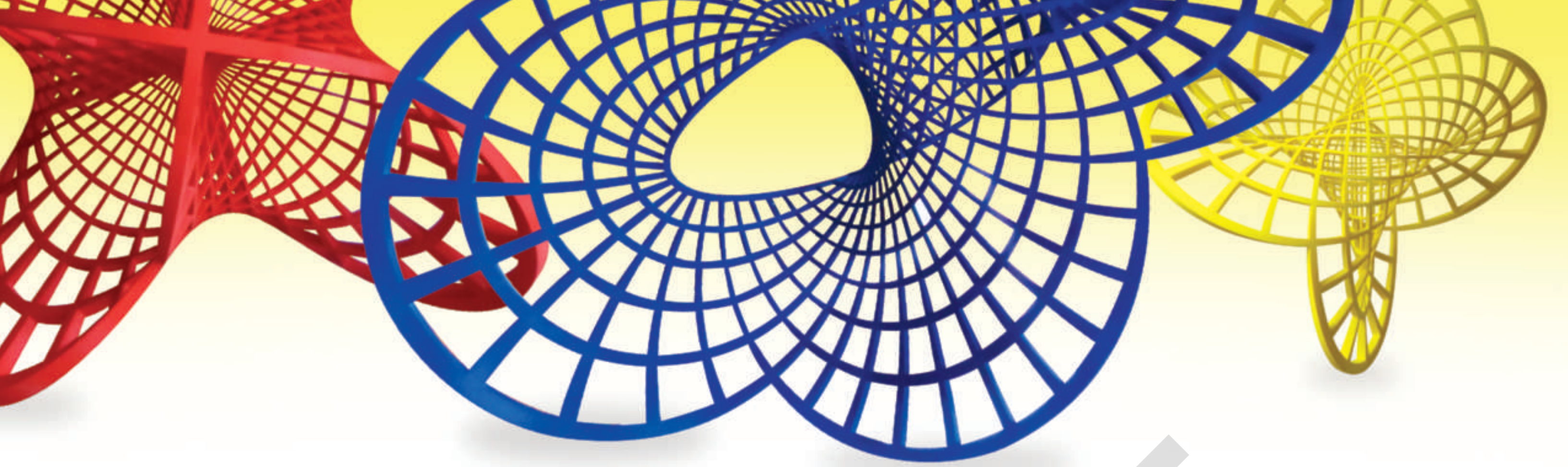


CHAPTER 14

Algebraic Thinking, Equations, and Functions

LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

14.1 Summarize each area of algebraic thinking

14.2 Describe connections between number and algebraic thinking.

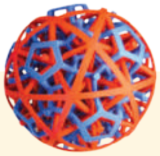
14.3 Explore ways to engage students in applying properties of the operations to number and algebra.

14.4 Illustrate and describe patterns and functions and describe how to engage students in learning about functions in K–8.

14.5 Analyze challenges students have with symbols (e.g., equal sign, variables) and identify strategies that can avoid or undo these limited conceptions.

14.6 Define mathematical modeling and describe ways to infuse modeling into instruction across mathematics.

Algebraic thinking (also called algebraic reasoning) begins in Kindergarten as young students "represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations" (CCSSO, 2010, p. 11) Similar connections between arithmetic and algebra are noted in every grade from kindergarten through fifth grade, where number and algebra are combined in the discussions of clusters and standards under the domain of Operations and Algebraic Thinking. In middle school, students begin to study algebra in more abstract and symbolic ways, focusing on understanding and using variables, expressions, and equations. CCSS-M introduces ***functions*** as a domain in grade 8, but ***functional thinking*** begins in the early grades as students consider situations that co-vary, such as the relationship between number of T-shirts purchased and the cost of those T-shirts. Algebraic thinking is present across content areas and is central to mathematical reasoning, as can be seen from the strong connections to the Standards for Mathematical Practice.



BIG IDEAS

- Algebra is a useful tool for generalizing arithmetic and representing patterns in our world. Explaining the regularities and consistencies across many problems gives students the chance to generalize (Mathematical Practice 8).

- The methods we use to compute and the structures in our number system can and should be generalized. For example, the generalization that $a + b = b + a$ tells us that $83 + 27 = 27 + 83$ without the need to compute the sums on each side of the equal sign (Mathematical Practice 7).
- Symbols, especially those involving equality and variables, must be well understood conceptually for students to be successful in mathematics (Mathematical Practice 6).
- The understanding of functions is strengthened when they are explored across representations (e.g., equations, tables, and graphs). Each mathematical model provides a different view of the same relationship (Mathematical Practice 4).

Strands of Algebraic Thinking

It is human to try to make sense of the world (and to see how things are related) (Fosnot & Jacob, 2010). Algebraic thinking is used to generalize arithmetic, to notice patterns that hold true in algorithms and with properties, and to reason quantitatively about such things as whether expressions are equivalent or not. Algebra must be presented in a way that students see it is a useful tool for making sense of all areas of mathematics and real-world situations.

Algebraic thinking involves forming generalizations from experiences with number and computation and formalizing these ideas with the use of a meaningful symbol system. Far from being a topic with little real-world use, algebraic thinking pervades all of mathematics and is essential for making mathematics useful in daily life.

Researchers suggest three strands of algebraic reasoning, all infusing the central notions of generalization and symbolization (Blanton, 2008; Kaput, 2008):

1. The study of structures in the number system, including those arising in arithmetic.
2. The study of patterns, relations, and functions.
3. The process of mathematical modeling, including the meaningful use of symbols.

Algebra is a separate strand of the mathematics curriculum but is also embedded in all areas of mathematics. This chapter is organized around these three strands, though we split 1 and 3 into two parts. In each section we share how these ideas develop across grades K–8.

Structure in the Number System: Connecting Number and Algebra

Algebra is often referred to as generalized arithmetic. For students to generalize an operation or pattern, they must look at several examples and notice what is happening in the problem, in other words, gain insights into the structure of the number system. Here we share three ways to connect arithmetic to algebra.

Number Combinations

Looking for generalizations in sets of problems can begin in the early grades, with decomposing numbers (CCSS-M K.NBT.A.1) and continue through the early grades as students use strategies to add and subtract (CCSS-M 1.OA.B.3 and 2.OA.B.2). The following task, based on Neagoy (2012), uses the context of birds to focus on decomposing numbers.

Seven birds have landed in your backyard, some landed on a tree, and some are at your feeder. How many birds might be in the tree and how many at the feeder?

Pause & Reflect

If there are 2 birds at the feeder, how many are in the tree? If there are 6 birds at the feeder how many birds are in the tree? How can you find how many in the tree if you know how many birds are at the feeder?

Generalizations can be analyzed when data is recorded in a table. Visit the companion website for the Birds in the Backyard Activity Page. Students can describe the pattern with symbols, something that elementary students can do and that middle school students must do. In CCSS-M variables are introduced in grade 3 and 4 as they solve for missing unknowns (3.OA.D.8 and 4.OA.A.3), and are emphasized across grades 6–8 in the Expressions and Equations Domain. For example, in the Birds in the Backyard problem, after students answer the questions in the Pause and Reflect, ask "If I have t birds in the tree, how might you describe how many birds are at the feeder?" Students might answer, "Seven minus t." Write $7 - t$. If the students answer "f" (for the variable to represent the number of birds at the feeder), then ask how t and f are related in an equation. Three equations could describe this situation: $t + f = 7$, $7 - f = t$, and $7 - t = f$.

Place-Value Relationships

Fundamental to mental mathematics is generalizing place-value concepts. Consider the sum 49 + 18. How would you add it in your head? Many people will move one over to make a 10 and think 50 + 17 (or move two over to get 47 + 20). Many of these strategies have been addressed in the previous four chapters.

The hundreds chart helps students to generalize the relationship between tens and ones. Ask students, "What did we add to get from 72 to 82? From 5 to 15? From 34 to 44?" Students notice across these examples (and more like them) that they are adding 10 and moving down exactly one row. Moves on the hundreds chart can be represented with arrows (for example, → means right one column or plus 1, and ↑ means up one row or less 10). Consider asking children to complete these problems:

14 →→←←	**63 ↑↑↓↓**	**45 →↑←↓**

Some students may count up or back using a count-by-ones approach. Others may jump 10 or 1 (up or down). Still others may recognize that a downward arrow "undoes" an upward arrow– an indication that these children are moving toward generalizations (Blanton, 2008). In other words, they recognize that +10 and −10 results in a zero change. Students can also write the equations for the arrow moves with numbers or with variables—for example, for the first problem, $n + 1 + 1 - 1 - 1 = n + (1 - 1) + (1 - 1) = n$. The more variables and numbers are used in looking at generalizations, the better able students become at using symbols.

CCSS Standards for Mathematical Practice

MP4. Model with mathematics.

***FORMATIVE ASSESSMENT* Notes.** As students work on such tasks, you can observe while using a **checklist** to note which students are solving by counting by ones, by jumping, or by noticing the "doing" and "undoing." What you observe can help your discussion as you can have students start by first sharing the more basic strategies and then have students who have generalized the situation share how they think about it.

Activity 14.1 provides an engaging context for students to explore patterns involving place value and addition.

Activity 14.1

CCSS-M: 1.NBT.C.4; 2.NBT.B.9; 3.OA.D.9

Diagonal Sums

STUDENTS *with* SPECIAL NEEDS

Provide each student with a hard copy of a Hundreds Chart, Blackline Master 3, or you can use an interactive virtual hundreds chart (numerous options are available on the Web). Students select any four numbers in the hundreds chart that form a square. Ask students to add the two numbers on each diagonal as in the example shown here. For younger students or students who struggle, use calculators so that they can explore the pattern without getting bogged down in computations.

13	14
23	24

Have students share their diagonal sums with a partner. Compare what happened. Together invite students to find another square. Ask them to describe why this pattern works. Allow time for each pair to share why the pattern works. To extend this activity, use diagonals of rectangles; for example, the numbers 15, 19, 75, and 79.

Pause & Reflect

Before reading further, stop and explore why the diagonal sums described in the previous activities are the same. What questions might you ask students to be sure they are noticing the relationship between tens and ones?

Here are some additional tasks you might explore on a hundreds chart. With each one, notice how the first aspect of the task is about number and the latter questions focus on generalizations (algebraic thinking).

- Pick a number. Move down two and over one. What is the relationship between the original number and the new number? What algebraic expression describes this move?
- Pick a number. Add it to the number to its left and to its right. Divide by 3. What answer do you get? Can you explain why this works? Can you explain using variables?
- Skip count by different values (e.g., 2, 4, 5). Which numbers make diagonal patterns? Which make column patterns? What is true about all numbers that make a column pattern?
- Find two skip-count numbers in which one number lands “on top of” the other (that is, all of the shaded values for one pattern are part of the shaded values for the other)? How are these two skip-count numbers related?

Asking questions such as “When will this be true?” and “Why does this work?” requires children to generalize and consequently, strengthen their understanding of the number concepts they are learning.

Algorithms

When studying the operations, students are often asked to explain how they solved a problem, for example, 504–198. As you listen to students’ strategies, record their ideas using symbols. For this problem, you might record a student’s mental strategy like this: $504 - 200 + 2$. Ask, “Does this show how you solved it?" or "Is this equivalent to the original expression?" Such questions to the class can lead to rich discussions about the properties (Blanton et al., 2011).

Slight shifts in how arithmetic problems are presented can open up opportunities for generalizations (Blanton, 2008). For example, instead of a series of unrelated computation problems, consider a list that can lead to a discussion of a generalization:

$$3 \times 7 = ? \qquad 6 \times 7 = ?$$

In discussing the relationship, students notice that 6 is twice 3 and therefore the answer is twice as much. This strategy can be applied to any of the $\times$ 6 basic facts, helping students with what is often some of the more challenging facts to learn.

Sets of problems are good ways for students to look for and describe patterns across the problems, patterns that build an understanding for the operation and related algorithms:

Standards for Mathematical Practice

MP8. Look for and express regularity in repeated reasoning.

$$\frac{1}{2} \times 12 = \qquad \frac{1}{4} \times 12 = \qquad \frac{1}{8} \times 12 = \qquad \frac{3}{4} \times 12 = \qquad \frac{3}{8} \times 12 =$$

Once students have solved sets of related problems, focus attention on what you want students to generalize: What do you notice? How are the problems alike? Different? How does the difference in the problem alter the answer? Why? In this set of problems, such a discussion can help students understand the relationship between the numerator and the denominator and what that means in multiplication situations.

Complete Self-Check 14.1: Structure in the Number System: Connecting Number and Algebra

Structure in the Number System: Properties

The importance of the properties of the operations cannot be overstated. Table 14.1 provides a list of the ones students must know, including how students might describe the property. In the CCSS-M, properties of the operations are one of the only things mentioned in the standards for each grade, grades 1 to 8. Importantly, the emphasis is on *using* and *applying* the properties (not identifying them); for example, in grade 2, one standard states, "Explain why addition and subtraction strategies work, using place value and the properties of operations" (CCSSO, 2010, p. 19).

The properties are essential to computation (Blanton, Levi, Crites, & Dougherty, 2011). Traditionally, instruction on the properties has been on matching equations to which property they illustrate. That is not sufficient and should not be the focus of your instruction on the properties. Instead, focus on helping students recognize and understand these important generalizations— and use them to generate equivalent expressions in order to solve problems efficiently and flexibly (for example, understanding the commutative property for both addition and multiplication reduces substantially reduces the number of facts to be memorized). The properties of the operations have been discussed in several chapters in this book, in particular Chapter 9 and Chapter 10. In this chapter the focus is on exploring the properties as generalizations of number.

Making Sense of Properties

Students begin to notice equivalent expressions as they engage in their work with numbers. For example, young students might be asked to solve this pair of equations: $3 + 7 =$ ____ and $7 + 3 =$ ____ . Some students may say that these equal the same, but it is important to discuss why this is true. A young student might explain that it could be beans on two plates and you moved the plates to trade places. Using or applying this same property means that in an

TABLE 14.1 PROPERTIES OF THE NUMBER SYSTEM

Properties of the Operations		
Name of Property	**Symbolic Representation**	**How a Student Might Describe the Pattern or Structure**
Addition		
Commutative	$a + b = b + a$	"When you add two numbers in any order you will get the same answer."
Associative	$(a + b) + c = a + (b + c)$	"When you add three numbers you can add the first two and then add the third or add the last two numbers and then add the first number. Either way you will get the same answer."
Additive Identity	$a + 0 = 0 + a = a$	"When you add zero to any number you get the same number you started with."
	$a - 0 = a$	"When you subtract zero from any number you get the number you started with."
Additive Inverse	$a - a = a + (-a) = 0$	"When you subtract a number from itself you get zero."
Inverse Relationship of Addition and Subtraction	If $a + b = c$ then $c - b = a$ and $c - a = b$	"When you have a subtraction problem you can 'think addition' by using the inverse."
Multiplication		
Commutative	$a \times b = b \times a$	"When you multiply two numbers in any order you will get the same answer."
Associative	$(a \times b) \times c = a \times (b \times c)$	"When you multiply three numbers you can multiply the first two and then multiply the answer by the third or multiply the last two numbers and then multiply the answer by the first number. Either way you will get the same answer."
Multiplicative Identity	$a \times 1 = 1 \times a = a$	"When you multiply one by any number you get the same number you started with."
Multiplicative Inverse	$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$	"When you multiply a number by its reciprocal, you will get one."
Inverse Relationship of Multiplication and Division	If $a \times b = c$ then $c \div b = a$ and $c \div a = b$	"When you have a division problem you can 'think multiplication' by using the inverse."
Distributive (Multiplication over Addition)	$a \times (b + c) = a \times b + a \times c$	"When you multiply two numbers, you can split one number into two parts (7 can be 2 + 5), multiply each part by the other number, and then add them together."

equation such as $394 + 176 = n + 394$, a student recognizes the commutative property and uses it to efficiently solve the problem. She may say that n must be 176 because $394 + 176$ is the same as $176 + 394$.

To ensure the property is generalized, ask students to share their generalization in symbols (e.g., $a + b = b + a$). This makes the connection from number to algebra explicit, while helping students understand how to write the properties symbolically. Using such letters to describe the properties can be used as early as first grade (Blanton et al., 2011; Carpenter, Franke, & Levi, 2003).

Just as sets of tasks can be used to generalize an algorithm, sets of tasks can be used to focus on the properties:

$\begin{array}{r} 35 \\ \underline{\times 52} \end{array}$	$\begin{array}{r} 52 \\ \underline{\times 35} \end{array}$	$\begin{array}{r} 23 \\ \underline{\times 46} \end{array}$	$\begin{array}{r} 46 \\ \underline{\times 23} \end{array}$
$\frac{1}{6} \times 12 =$	$12 \times \frac{1}{6} =$	$\frac{2}{3} \times 12 =$	$12 \times \frac{2}{3} =$

CCSS Standards for Mathematical Practice

MP7. Look for and make use of structure.

Although students may understand the commutative property of multiplication with whole numbers, they may not recognize that the property also applies to fractions (in fact, all real numbers). Ask students "Is this true for fractions?" "Is it true for other types of numbers?" "All numbers?" Activity 14.2 provides a creative way for students to understand the identity for addition and/or multiplication, as well as other properties.

Activity 14.2 CCSS-M: 1.OA.B.3; 1.OA.C.6; 1.OA.D.7; 2.OA.B.2; 3.OA.D.9

Five Ways to Zero

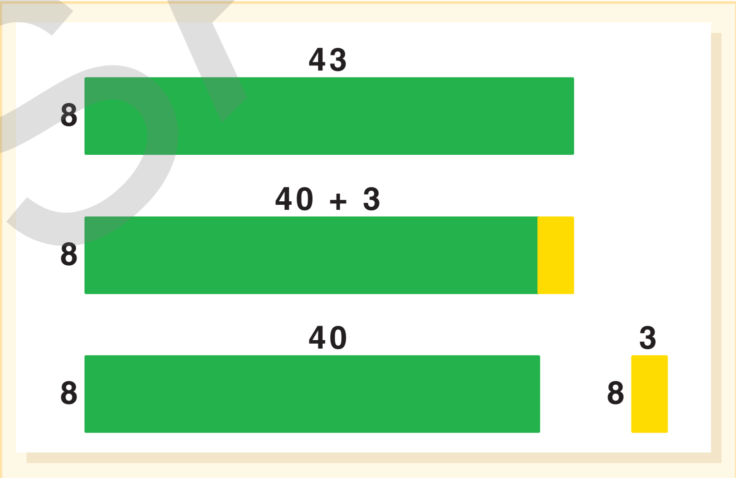
Place students in partners. Give each pair a number (you can use a deck of cards). If they get a 7, they are to write 5 different ways to get to 0 using number sentences. For example, they could write 7−5−2 or it could be 7 + 3−10. Be sure students are using correct notation and grouping so that their statements are true. As a follow-up, ask students to find five ways to get their own number as the answer. All students might benefit from using counters or a number line in exploring the possibilities. After they find five ways, ask students what was true about all of the problems they wrote.

Discussing problem sets such as these, and others, helps students to make sense of the properties. In this first grade vignette, the teacher is helping students to reason about the commutative property.

Teacher: [*Pointing at* $5 + 3 = 3 + 5$ *on the board.*] Is it true or false?
Carmen: True, because $5 + 3$ is 8 and $3 + 5$ is 8.
Andy: There is a 5 on both sides and a 3 on both sides and nothing else.
Teacher: [*Writing* $6 + 9 = 9 + 6$ *on the board.*] True or false?
Class: True! It's the same!
Teacher: [*Writing* $25 + 48 = 48 + 25$ *on the board*] True or false?
Children: True!
Teacher: Who can describe what is going on with these examples?
Rene: If you have the same numbers on each side, you get the same thing.
Teacher: Does it matter what numbers I use?
Class: No
Teacher: [*Writing* $a + 7 = 7 + a$ *on the board*] What is a?
Michael: It can be any number because it's on both sides.
Teacher: [*Writing* $a + b = b + a$ *on the board]* What are a and b?
Children: Any number!

Notice how the teacher is developing the commutative property of addition in a conceptual manner—focusing on exemplars to guide students to generalize, rather than asking them to memorize or identify the properties, which can be a meaningless, rote activity.

The structure of numbers can sometimes be illustrated geometrically. For example, 8×43 can be illustrated as a rectangular array. That rectangle can be partitioned (symbolically this is $8(40 + 3)$ and then represented as two rectangles [e.g., $(8 \times 40) + (8 \times 3)$], preserving the quantity:

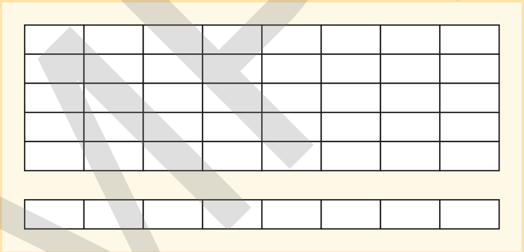


Challenge students to think about this idea *in general*. This may be described in words (at first) and then as symbols: $a \times b = (c \times b) + (d \times b)$, where $c + d = a$. Be sure students can connect the examples to general ideas and the general ideas back to examples. This is the distributive property, and it is perhaps the most important central idea in arithmetic (Goldenberg, Mark, & Cuoco, 2010).

Applying the Properties of Addition and Multiplication

Noticing generalizable properties and attempting to prove that they are true is a significant form of algebraic reasoning and is at the heart of what it means to do mathematics (Ball & Bass, 2003; Carpenter et al., 2003; Schifter, Monk, Russell, & Bastable, 2007). Students can make conjectures about properties as early as first or second grade, and this must be encouraged by the teacher. For example, as first graders are exploring addition, a student might suggest that the order you add two numbers does not matter ($1 + 6 = 6 + 1$). This idea can be presented to the group. Students can test the idea with other numbers and eventually try to explain why this is true. They can write it in symbols: $a + b = b + a$ (yes, first graders can understand and benefit from using variables). It is important to ask, how can this help us do math problems? For a first grader, recognizing that when they see $1 + 6$ that they can start with 6 and add on 1 is an important step toward learning how to add efficiently (as well as learn the basic facts).

The distributive property is central to basic facts and the algorithms for the operations. For example, in learning the multiplication facts, students learn derived fact strategies. For 6×8, they can split up the 6 however they like, multiply its parts by 8 and then put it back together (e.g., $(5 \times 8) + (1 \times 8) = 6 \times 8$). They may justify this with the following rectangles:



Beyond the properties in Table 14.1, students can apply the properties to make other conjectures, the focus of Activities 14.3 and 14.4.

Activity 14.3

CCSS-M: 1.OA.B.3; 2.OA.C.3; 2.NBT.B.5; 3.OA.D.9

Broken Calculator: Can You Fix It?

Distribute calculators to every student. In partners, have students select one of these two problems to explore. They must decide if it is possible, share an example of how to do it (if it is possible), and finally prepare a justification or illustration to describe why it does or doesn't work (in general).

1. If you cannot use any of the even keys (0, 2, 4, 6, 8), can you create an even number in the calculator display? If so, how?
2. If you cannot use any of the odd keys (1, 3, 5, 7, 9), can you create an odd number in the calculator display? If so, how?

Invite early finishers to take on the other problem or to write their justifications using variables. In the follow-up discussion, ask students for other patterns or generalizations they notice about odd and even numbers.

Activity 14.4

CCSS-M: 1.OA.B.3; 2.NBT.B.9; 3.OA.B.5; 5.OA.A.1

Convince Me Conjectures

ENGLISH LANGUAGE LEARNERS

STUDENTS *with* SPECIAL NEEDS

To begin, offer students a conjecture to test (see Conjecture Cards Activity Page for ideas for the four operations). For example, "If you add one to one addend and take one away from the other addend, the answer will be the same." Ask students to (1) test the conjecture and (2) prove it is true for any number. Point out the difference between testing and proving. Then, invite students to create their own conjecture (in words) that they believe is always true. Then they must prepare a visual or explanation to convince others that it is true. All students, particularly ELLs, may struggle with correct and precise terms. You can "revoice" their ideas using appropriate phrases to help them learn to communicate mathematically, but be careful to not make this the focus—the focus should be on the ideas presented. Students with disabilities benefit from the presentation and discussion of counterexamples.

Odd and even numbers provide an excellent context for exploring structure of the number system. Students will often observe that the sum of two even numbers is even, that the sum of two odd numbers is even, or that the sum of an even and an odd number is always odd. To explain why two odds make an even, a student might explain that when you divide an odd number by two, there will be a leftover. If you do this with the second odd number, it will have a leftover also. The two leftovers will go together so there won't be a leftover in the sum. Students should also use manipulatives such as connecting cubes to show their reasoning.

Figure 14.1 shows a fourth grader's "Convince Me Conjecture" illustration for the conjecture: *For a product, you can take half of the one factor and double the other factor and you will get the same answer.* Although the student's work only shows two examples, the student is illustrating that this process of cutting the array and moving it down below, can be generalized for other numbers. Ask students how they might write the conjecture in symbols. For the example above, students might write "If $a \times b = c$, then $\frac{1}{2}a \times 2b = c$." In addition, the Expanded Lesson One Up, One Down focuses on a conjecture, starting with trying out numbers, which then generalizing using tools or reasoning (see discussion in Chapter 2 for this activity).

Using and *applying* the properties is central to mathematical proficiency—it is not only emphasized in the CCSS-M content, but also in the Mathematical Practices (CCSSO, 2010). An explicit focus on seeking generalizations and looking for structure is also important in supporting the range of learners in the classroom, from those who struggle to those who excel (Schifter, Russell, & Bastable, 2009). Doing so requires planning—deciding what questions you can ask to help students think about generalized characteristics within the problems they are doing—across the mathematical strands (not just when they are in an "algebra" unit).

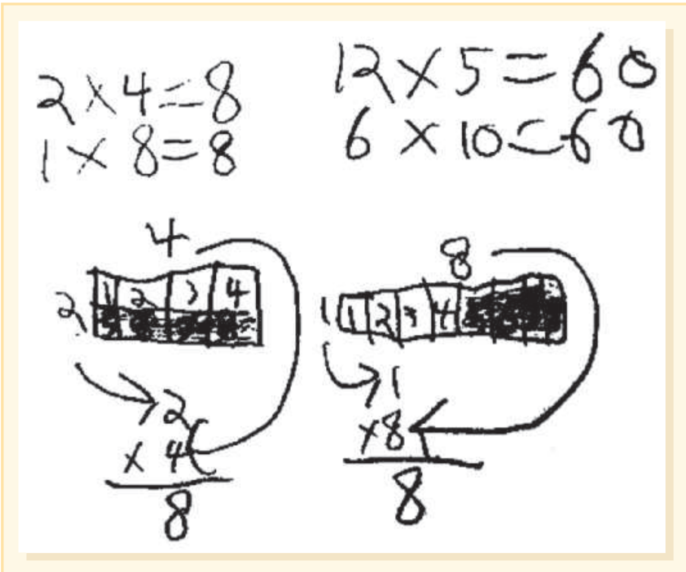


FIGURE 14.1 A student justifies a conjecture.

Complete Self-Check 14.2: Structure in the Number System: Properties

Study of Patterns and Functions

Patterns are found in all areas of mathematics. Learning to look for patterns and how to describe, translate, and extend them is part of thinking algebraically. Two of the eight Standards for Mathematical Practice begin with the phrase "look for," implying that students who are mathematically proficient pay attention to patterns as they do mathematics. Functional thinking begins in pre-K–2 when students make observations like, "Each person we add to the group we add 2 more feet" (Blanton et al., 2011). Let's look at a modified version of the *Birds in the Backyard* problem:

Five birds have landed in your backyard, some at the feeder and some in the tree. How many ways might the birds be in the tree and at the feeder?

Pause & Reflect

Can you list all the ways in which this can occur with 5 birds? With 7 birds? With 10 birds? Is there a rule to describe the number of ways in which 10 birds can be sitting in the tree and at the feeder?

There are 6 ways for 5 birds to be in the tree and at the feeder, 8 ways for 7 birds, and 11 ways for 10 birds. Upper elementary and middle school students should be able to explain why this is the case: For any number of birds (n), there are $n + 1$ ways because there can be 0, 1, 2, n birds at the feeder. Using a problem that is concrete and that begins with listing numeric possibilities is a way to help students learn to generalize a pattern. To extend the discussion, ask students questions such as these: "What if there were 340 birds? Would the rule still hold? If you knew there were 20 different ways in which the birds could be in the backyard, how many birds are there? Is there a rule for that?"

The study of patterns and functions is infused throughout the CCSS-M, and explicitly addressed in these standards:

- Identify arithmetic patterns (https://www.youtube.com/watch?v=1X-RWOTsHQw&list=PLnIkFmW0ticNxpCDlb7vOE7j4qrZtXrX8) (including patterns in the addition table or multiplication table). (grade 3)
- Generate a number or shape pattern that follows a given rule. (https://www.youtube.com/watch?v=vKcXncMARu4&list=PLnIkFmW0ticOLvcMb6M3YyFBsGRMB_zLs) Identify apparent features of the pattern that were not explicit in the rule itself. (grade 4)
- Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. (grade 5)
- Represent and analyze quantitative relationships between dependent and independent variables. (grade 6)
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations. (grade 7)
- Understand the connections between proportional relationships, lines, and linear equations. (grade 8)
- Define, evaluate, and compare functions. (https://www.youtube.com/watch?v=Jq-1RjEIulA) (grade 8)
- Use functions to model relationships between quantities. (grade 8)

This list is an indication of the importance of focusing on patterns and functions for many years before students reach the formal algebra course.

Repeating Patterns

Repeating patterns are those patterns that have a core that repeats. For example, red-blue could the core and a string of beads continues to repeat this pattern: red-blue-red-blue-red. . . . Repeating patterns is not mentioned in the CCSS-M, but looking for patterns in operations and in numbers (place value) is found across the grades. Non-number patterns can build a foundation for later noticing odd-even-odd-even patterns. Physical materials provide a trial-and-error approach and allow patterns to be extended beyond the few spaces provided on a page. By using a variety of materials such as colored blocks, buttons, and connecting cubes to create and extend their patterns, students begin to generalize ideas of patterns. These can be recorded symbolically, for example red-blue is an AB pattern because the core has two different elements, A and B. Figure 14.2 provides some illustrations.

An important concept in working with repeating patterns is for students to identify the *core* of the pattern (Warren & Cooper, 2008). One possible way to emphasize the core is to place shape patterns under a document camera, say aloud what is there and ask what comes next. After a few add-ons, ask students what the pattern is. Label the pattern with letters (i.e., an ABC pattern has three different shapes that repeat).

Repeating patterns are everywhere! The seasons, days of the week, and months of the year are just a beginning. Ask students to think of real-life AB patterns—for example, "to school, home from school" or "set table before eating, clear table after eating." Children's books often have patterns in repeating rhymes, words, or phrases. *Pattern Fish* (Harris, 2000) and *My Mom and Dad Make Me Laugh* (Sharratt, 1994) are two great choices. A very long repeating pattern can be found in *If You Give a Mouse a Cookie* (Numeroff, 1985) (or any of this series), in which each event eventually leads back to giving a mouse a cookie, with the implication that the sequence would be repeated.

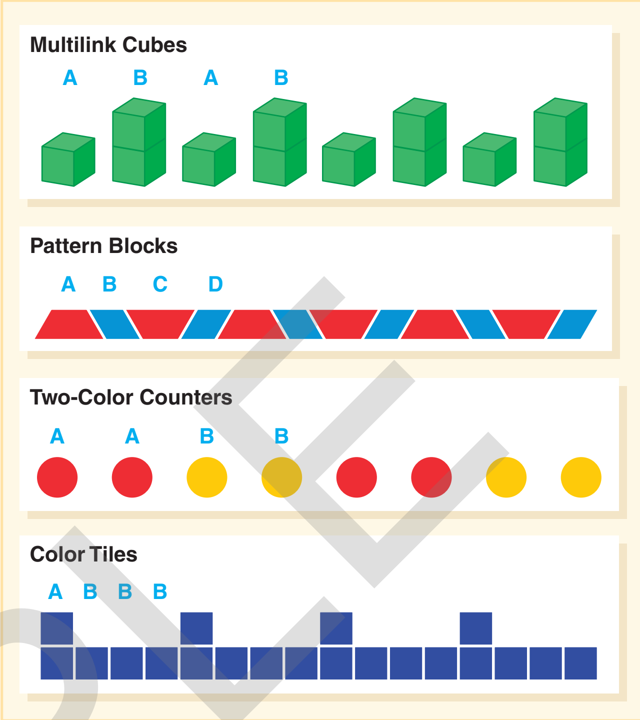


FIGURE 14.2 Examples of repeating patterns with various manipulatives.

Oral patterns can be recited. For example, "do, mi, mi, do, mi, mi" (ABB) is a simple singing pattern. Body movements such as waving the arm up, down, and sideways can make patterns: up, side, side, down, up, side, side, down (ABBC). There are numerous sites on the Web for exploring repeating patterns. For example, NLVM has several explorations with repeating (and growing) patterns, including Attribute Trains, Block Patterns, Color Patterns, Pattern Blocks, and Space Blocks. Repeating patterns can be used to strengthen students understanding of number, for example multiples and division with remainders, as in Activity 14.5.

Activity 14.5

CCSS-M: 2.OA.C.3; 4.OA.B.4

Predict Down the Line

Provide students with a pattern to extend (e.g., ABC pattern made with colored links). Before students begin to extend the pattern, have them predict exactly what elements (links) will be in, say, the twelfth position. (Notice that in an ABC pattern the third, sixth, ninth, and twelfth terms are the C element because they are multiples of 3.) After students predict, have them complete the pattern to check. Ask them how they knew. You can differentiate this lesson by starting with a more basic AB pattern or by having different groups of students work on different types of patterns based on their readiness. Students should be required to provide a reason for their prediction in writing supported with visuals.

In fourth grade, connect to the idea of remainders connecting the repeating patterns to division. Ask students to predict the 100th term for ABC pattern. Because $100 \div 3 = 33$ remainder 1, it would be the A element in the pattern.

Predicting has some interesting real-world contexts appropriate for upper elementary and middle school students. One context is the Olympics (Bay-Williams & Martinie, 2004b). The Summer Olympics are held in 2016, 2020, and every four years after that. The Winter Olympics are held in 2014, 2018, and so on. This pattern can be described using variables.

The years with Summer Olympics are in the form $4n$ and Winter Olympics in the form $4n + 2$. Hurricanes, the focus of Activity 14.6, are also in a repeating pattern (Fernandez & Schoen, 2008).

Activity 14.6

CCSS-M: 4.OA.A.3; 4.OA.B.4; 5.OA.B.3; 6.EE.B.2a

Hurricane Names

Ask students what they know about hurricanes and hurricane names. Hurricanes are named such that the first one of the year has a name starting with A, then B, and so on. For each letter, there are six names in a six-year cycle using an ABCDEF pattern (except those that are retired when a major hurricane has that name). The gender of the names alternate in an AB pattern. Invite students to select a letter of the alphabet and look up the list of six names. Ask students to answer questions such as these (assume the names do not get retired):

- **What years will the hurricanes be named after the first name on your list? The last name? What years will the hurricanes be a girl's name?**
- **What will the hurricane's name be in the year 2020? 2030? 2050?**
- **Can you describe in words how to figure out the name of a hurricane, given the year?**

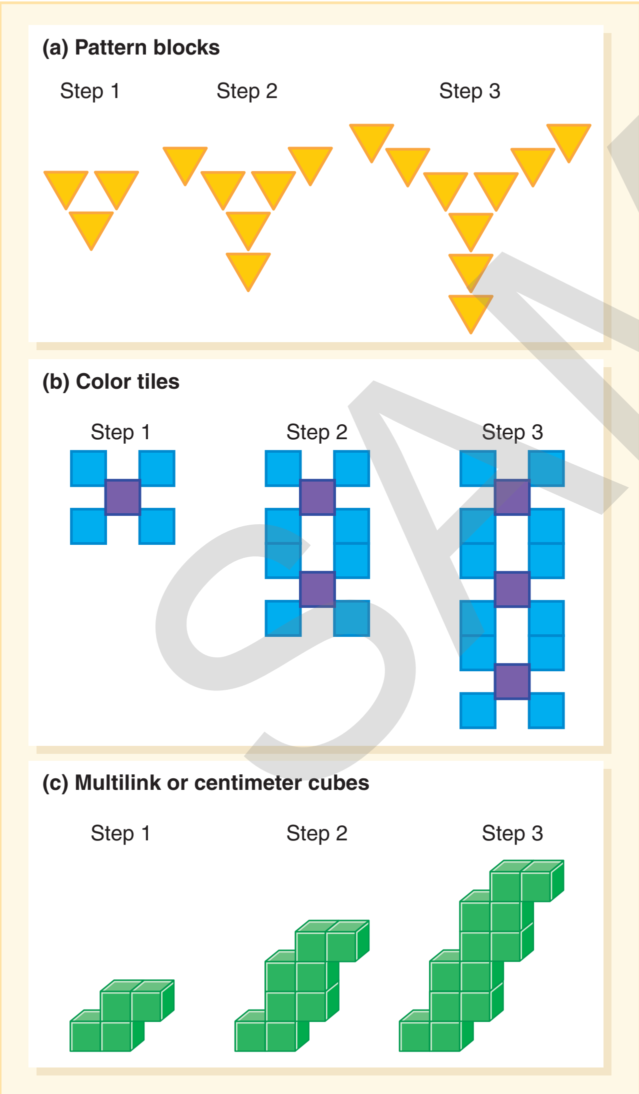


FIGURE 14.3 Geometric growing patterns using manipulatives.

Growing Patterns

Beginning in the primary grades and extending through the middle school years, students can explore patterns that involve a progression from step to step. In technical terms, these are called *sequences;* we will simply call them *growing patterns*. With these patterns, students not only extend or identify the core but also look for a generalization or an algebraic relationship that will tell them what the pattern will be at any point along the way (e.g., the nth term). Figure 14.3(a) is a growing pattern in which design 1 requires three triangles, design 2 requires six triangles, and so on—so we can say that the number of triangles needed is a function of which design it is (in this case, number of triangles $= 3 \times$ design number).

Geometric patterns make good exemplars because the pattern is easy to see and because students can manipulate the objects. Figure 14.3 shows three different growing patterns, though the possibilities for visuals and patterns are endless. The questions in Activity 14.7, mapped to the pattern in Figure 14.3(a), can be adapted to any growing pattern and help students begin to reason about functional situations.

Analyzing growing patterns should include the developmental progression of reasoning by looking at the visuals, then reasoning about the numerical relationships, and then extending to a larger (or nth) case (Friel & Markworth, 2009). Students' experiences with growing patterns should start with fairly straightforward patterns (such as in Figure 14.3) and continue with patterns that are more complicated (such as the Dot Pattern in Figure 14.4).

Activity 14. 7

CCSS-M: 5.OA.B.3; 6.EE.C.9; 7.EE.B.4a; 8.F.A.1

ENGLISH LANGUAGE LEARNERS

Predict How Many

Distribute Predict How Many Windows or Predict How Many Dot Arrays. Working in pairs or small groups, have students explore a pattern and respond to these questions:

- **Complete a table that shows the number of triangles for each step.**

Step Number	1	2	3	4	5 · · ·	10	20
Number of Triangles (Element)							

- **How many triangles are needed for step 10? Step 20? Step 100? Explain your reasoning.**
- **Write a rule (in words) that gives the total number of pieces to build any step number.**
- **Write a rule in symbols using the variable *n* for step number.**

See Expanded Lesson Exploring Functions through Geometric Growing Patterns for more details.

Keep in mind that ELLs need clarification on the specialized meanings of *step* and *table* because these words mean something else outside of mathematics.

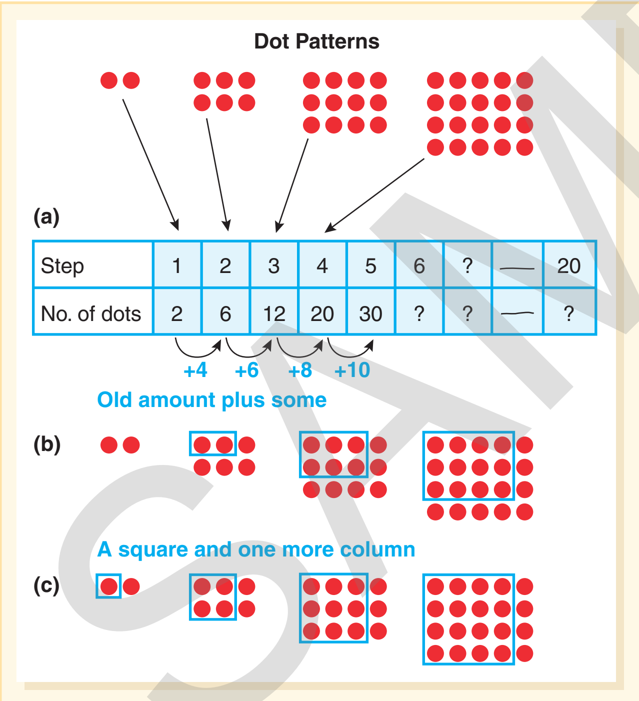


FIGURE 14.4 Analyzing relationships in the "Dot Pattern."

Term	1	2	3	4
Fraction	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$

If the list of fractions above continues in the same pattern, which term will be equal to 0.95?

Ⓐ The 100th
Ⓑ The 95th
Ⓒ The 20th
Ⓓ The 19th
Ⓔ The 15th

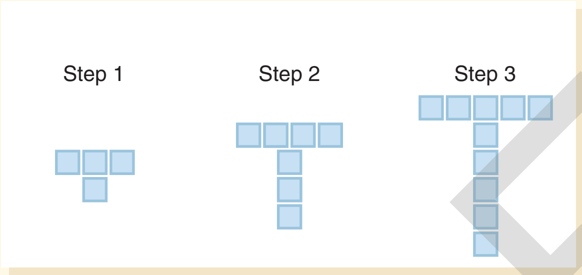
FIGURE 14.5 NAEP item for 13-year-olds.

Source: Lambdin, D. V., & Lynch, K. (2005). "Examining Mathematics Tasks from the National Assessment of Educational Progress." *Mathematics Teaching in the Middle School, 10*(6), 314–318. Reprinted with permission. Copyright © 2005 by the National Council of Teachers of Mathematics. All rights reserved.

It is also important to include fractions and decimals in working with growing patterns. In 2003, the National Assessment of Educational Progress (NAEP) tested 13-year-olds on the item in Figure 14.5. Only 27 percent of students answered correctly (Lambdin & Lynch, 2005).

Relationships in Functions

When students are exploring a growing pattern, there are three types of patterns they might notice. And, noticing the types of patterns is developmental (Blanton et al., 2011; Tanish, 2011). Each is shared here, connected to the growing T:



Step Number	Number of Tiles
x	y
1	4
2	7
3	10
4	?

Recursive Patterns. The description that tells how a pattern changes from step to step is known as a *recursive* pattern (Bezuszka & Kenney, 2008; Blanton, 2008). For most students, it is easier to see the patterns from one step to the next, seeing the increase (or decrease). For the T pattern, this is noticing that the right column goes up by 3 each time. In the Dot Pattern (Figure 14.4(a)), the recursive pattern is adding successive even numbers.

The recursive pattern can also be observed in the physical pattern and in the table. In the T pattern, we see that one tile is added to the top of the T and two tiles to the stem. In the dot Pattern, notice that in each step, the previous step has been outlined (Figure 14.4(b)).

CCSS **Standards for Mathematical Practice**

MP2. Reason abstractly and quantitatively.

Covariational Thinking. Covariational thinking involves noticing how two quantities vary in relation to each other and being explicit in making that connection (Blanton et al., 2011). In the T pattern, a student might say, "As the step number grows by 1, the number of tiles needed goes up by 3." Notice that developmentally this is more developmentally sophisticated than just noticing a skip pattern, as the student is connecting how the change in one quantity affects the change in the other quantity (i.e., how they covary). Students in grades 3–5 can employ covariational thinking (Tanish, 2011), and doing so is important in building their algebraic thinking.

Correspondence Relationship. A *correspondence relationship* (also known as the *explicit rule*) is a correlation between two quantities expressed as a function rule (Blanton et al., 2011). In other words, it is begin able to look across the table to see how to use the input (x) to generate the output (y). In the T pattern, the rule is $3x + 1$. Imagine that you needed to find the number of tiles needed for the hundredth T. If you use recursive thinking, you will need to find all of the prior 99 entries in the table. If you notice how x and y correspond (the explicit rule), you can use that rule to find how many tiles are needed for the hundredth T.

Pause & Reflect

Can you determine the correspondence relationship (explicit rule) for the dot pattern in Figure 14.4? Where do you see the relationship in the table? In the dots?

You might analyze the table and notice that if they multiply the step number by the next step number, they will get the number of dots for that step. This leads to the explicit rule or function: $d = n(n + 1)$, where d is number of dots and n is the step number. Or, you might analyze the dots to see what is changing. In Figure 14.4(c), a square array is outlined for each step. Each successive square is one larger on a side and the side of each square is the same as the step number. The column to the right of each square is also the step number. The numeric expression is $1^2 + 1$, $2^2 + 2$, $3^2 + 3$, and $4^2 + 4$. The explicit rule or function is therefore $d = n^2 + n$.

Input-output activities can begin with young children. The book *Two of Everything* (Hong, 1993) works well because the Haktaks put things "in the pot" and then take things "out of the pot." The book tells of the pot doubling, but that rule can be changed. Shoeboxes or large refrigerator boxes can be turned into input-output boxes. Decorate the box to look like a machine and add buttons for "easy," "medium," and "hard," and design functions that are appropriate for the grade of your students (Fisher, Roy, & Reeves, 2013). Virtual function machines can be found at NLVM, Math Playground, and Shodor Project Interactive, among others.

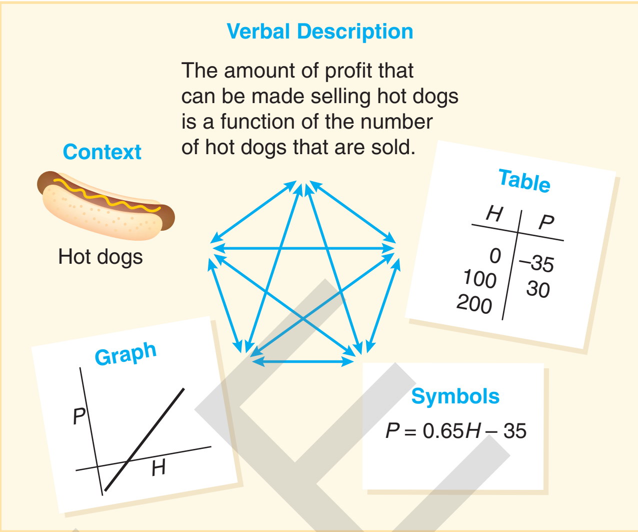


FIGURE 14.6 Five representations of a function for the situation of selling hotdogs.

Regardless of whether students use the table or the manipulatives, they will likely be able to describe the explicit formula in words before they can write it in symbols. If the goal of your lesson is to find the "rule," then stopping with the verbal formula is appropriate. In this case, you may have some students who are ready to represent the formula in symbols, and they can be challenged to do so as a form of differentiating your instruction. If your instructional goal is to write formulas using symbols, then ask students to first write the rule in words and then think about how they can translate that statement to numbers and symbols.

***TECHNOLOGY* Note.** There are several websites that focus on relationships in functions. NCTM's Illuminations website has a lesson titled, "The Crow and the Pitcher: Investigating Linear Functions Using a Literature-Based Model." PBS Kids' CyberChase has a fun game called "Stop That Creature" in which students figure out the rule that runs the game to shut down the creature-cloning machine. ■

Graphs of Functions

So far, growing patterns have been represented by (1) the physical materials or drawings, (2) a table, (3) words, and (4) symbols. A graph adds a fifth representation, and one that illustrates covariation. Figure 14.6 illustrates what these five representations look like with the context of selling hotdogs. Importantly, given any one of these representations, students need to be able to generate the others and understand how they are related.

Figure 14.7 shows the graph for the T pattern and the dot pattern. Notice that the first is a straight-line (linear) relationship and the other is a curved line that would make half of a parabola if the points were joined.

Activity 14.8

CCSS-M: 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2

Perimeter Patterns

Using a document camera or interactive white board, show rows of same-shape pattern blocks (see Figure 14.8). Working in pairs or small groups, have students build each pattern and explore what patterns they notice about how the perimeter grows. Ask: What is the perimeter of a row with 6 squares? 10 squares? Any number of squares? Repeat the process with trapezoids and hexagons (or have different groups of students working on different shapes). Distribute a Coordinate Graph, Blackline Master 22). Ask students to create a graph to illustrate the relationship between the number of pattern blocks in the row and the perimeter.

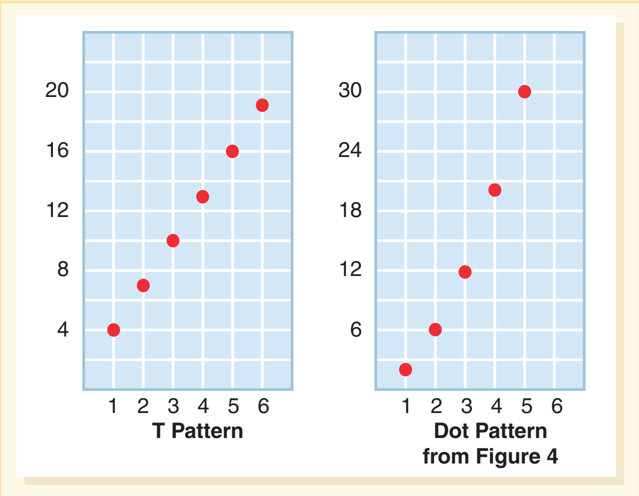


FIGURE 14.7 Graphs of two growing patterns.

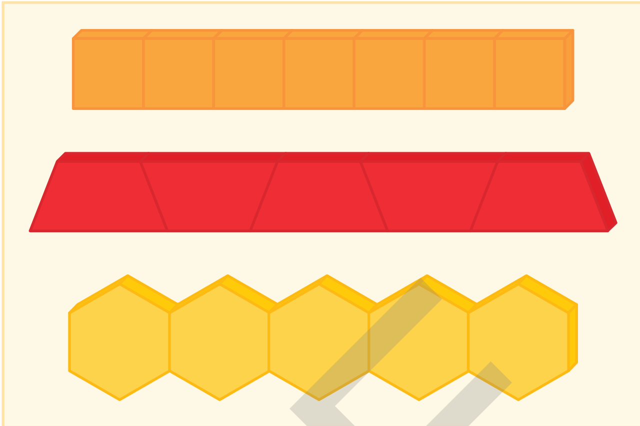


FIGURE 14.8 Same-color strings of pattern blocks. Can you determine the perimeter for *n* pattern blocks in a string?

Pause & Reflect

Which representations do you find most useful in determining the explicit rule or function? Which representation do you think students new to exploring patterns will use?

Having graphs of three related growing patterns offers the opportunity to compare and connect the graphs to the patterns and to the tables (see Figure 14.9). For example, ask students to discuss how to get from one coordinate to the next (e.g., up six, over one for the hexagon), and then ask how that information can be found in the table.

A graph encourages covariational thinking, which can lead to identifying the function. Pose the following questions to students to support covariational thinking and help students:

- How does each graph represent each of the row patterns?
- Why is there not a line connecting the dots?
- Why is one line steeper than the others?
- What does this particular point on the graph match up to in the model and the table?

FIGURE 14.9 Graphs of the perimeters of three different pattern-block strings.

***TECHNOLOGY* Note.** Function graphing tools permit users to create the graph of almost any function very quickly. Multiple functions can be plotted on the same axis. It is usually possible to trace along the path of a curve and view the coordinates at any point. The dimensions of the viewing area can be changed easily so that it is just as easy to look at a graph for x and y between -10 and $+10$ as it is to look at a portion of the graph thousands of units away from the origin. By zooming in on the graphs, it is possible to find points of intersection without algebraic manipulation or to confirm an algebraic manipulation. Similarly, the point where a graph crosses the axis can be found to as much precision as desired.

Digital programs can also be used for these purposes, and add speed, color, visual clarity, and a variety of other interesting features to help students analyze functions. Graphsketch is an online demonstration tool for making graphs of equations. For the teacher, Modeling Middle School Mathematics (MMSMath) offers video clips, such as "V-Patterns, Beans, Hair & Nails," where students explore patterns with formulas and represent solutions using linear equations, graphs, and tables. ■

FORMATIVE ASSESSMENT Notes. Being able to make connections across representations is important for understanding functions, and the only way to know if a student is seeing the connections is to ask. In a **diagnostic interview,** ask questions like the ones just listed and check to see whether students are able to link the graph to the context, to the table, and to the formula. ■

Students also need opportunities to estimate with graphs. Exploring situations without exact values focuses students' attention on covariation relationships, the focus of Activity 14.9.

Activity 14.9

CCSS-M: 6.EE.D.9; 8.F.B.5

Sketch a Graph

ENGLISH LANGUAGE LEARNERS

STUDENTS *with* SPECIAL NEEDS

Sketch a graph for each of these situations. No numbers or formulas are to be used.

a. The temperature of a frozen dinner from 30 minutes before it is removed from the freezer until it is removed from the microwave and placed on the table. (Consider time 0 to be the moment the dinner is removed from the freezer.)
b. The value of a 1970 Volkswagen Beetle from the time it was purchased to the present. (It was kept by a loving owner and is in top condition.)
c. The level of water in the bathtub from the time you begin to fill it to the time it is completely empty after your bath.
d. Profit in terms of number of items sold.
e. The height of a thrown baseball from when it is released to the time it hits the ground.
f. The speed of the same baseball.

Be sure that the contexts you pick are familiar to students, including ELLs. If they are not, change the context or illustrate what it is. Have students sketch their graphs without identifying which situation they selected (no labels on the graphs). Then display them on a projector. Let students examine the graph to see whether they can determine the matching situation.

Rather than sketch-a-graph, you can do match-a-graph (see illustrations in Figure 14.10). Match a graph can be a first experience as a scaffold, or as an alternative for students with disabilities.

Finally, it is also worthwhile to have students look at a graph and write a story to match (see Create a Journey Story Activity Page).

Describing Functions

Discrete and Continuous. Even in elementary school, the discussion of functions, especially graphical representations, should include a discussion of whether the points plotted on the graph should be connected or not and why. In the pattern blocks perimeter problem, the answer is no; the points should not be connected because you will only use whole-number values for counting blocks. When isolated or selected values are the only ones appropriate for a context, the function is *discrete*. If all values along a line or curve are solutions to the function, then it is *continuous*. The Sketch a Graph situations are all continuous, with the independent variable of time.

Domain and Range. The *domain* of a function comprises the possible values for the independent variable. If it is discrete, like the pattern blocks perimeter problem, it may include all positive whole numbers. For the 24-meter rectangular pen, the domain is all real numbers between 0 and 12.

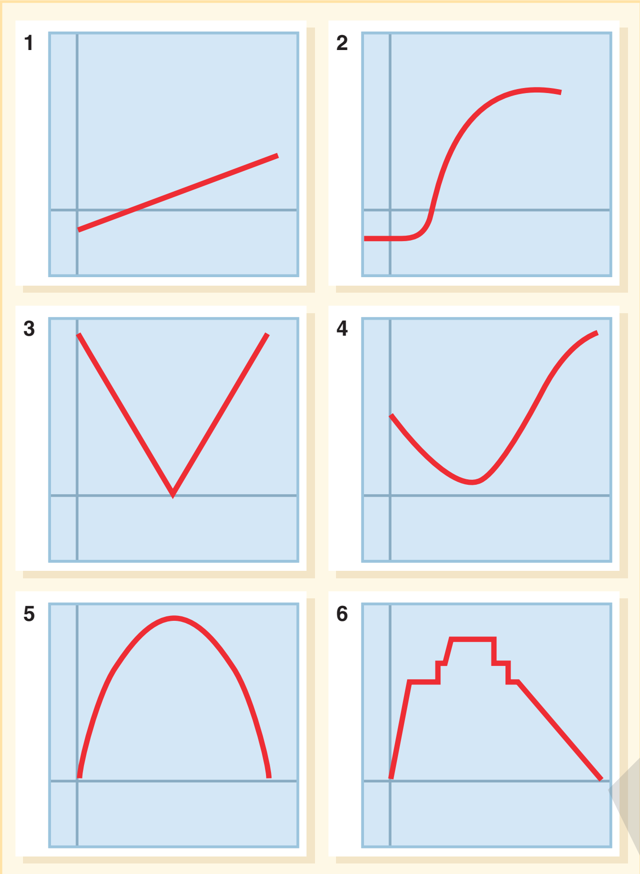


FIGURE 14.10 Match each graph with the situations described in Activity 14.9. Talk about what change is happening in each case.

The *range* is the corresponding possible values for the dependent variable. In the pattern blocks perimeter problem, the range is the positive whole numbers. In the rectangular pen, the range for the length is the same as the domain—real numbers between 0 and 12.

Linear Functions

Linear functions are a subset of functions, which can be linear or nonlinear. But because linearity is a major focus of middle school mathematics, and because growing patterns in elementary school tend to be linear situations, it appears here in its own section. *Curriculum Focal Points* and the Common Core State Standards emphasize the importance of linear functions across the middle grades, with a strong focus on linearity in seventh and eighth grade (CCSSO, 2010; NCTM, 2006).

Pause & Reflect

Think back to example tasks shared in this chapter. Which are linear functions? Which are not linear, but are still functions?

The examples that involved linear functions include the birds (how many ways they could be on the bush and in the tree), the geometric growing patterns, the T pattern, and the pattern block perimeters. The dot pattern is nonlinear (it is quadratic). For linear functions, the key is to focus on the idea that the recursive pattern has a constant rate of change—this is a central concept of linearity (Smith, 2008; Tanish, 2011).

In middle school, students are to notice if situations are linear or not (CCSSO, 2010). Consider the following example.

You are asked to build a rectangular pen with 24 yards of fence. (1) Write an equation to describe the relationship between the width and the area. (2) Write an equation to describe the relationship between the length and width.

An explicit formula for the width is $w = 12 - l$ (l is the length), which decreases at a constant rate, therefore looking like a line. By contrast, the explicit formula for area of the pen is $a = l(12 - l)$—it rises in a curve, reaches a maximum value, and then goes back down (see Figure 14.11).

Linear (and nonlinear) situations should be analyzed across representations (picture/objects, equation, graph, table, and situation). In a graph, this can be established by seeing that the plotted points lie on one line. In a table, the change will be constant (e.g., a recursive pattern is +4 each time). In the equation, linearity can be determined by looking at the part of the expression that changes and seeing if it represents constant change or not.

Rate of Change and Slope. Rate, whether constant or varying, is a type of change associated with how fast something is traveling. Rates can be seen in a wide range of contexts, such as the geometric model of the pattern block perimeters or the rate of growth of

Building a Pen

L	0	1	2	3	4	5	6	7	8	9	10	11	12
W	12	11	10	9	8	7	6	5	4	3	2	1	0
A	0	11	20	27	32	35	36	35	32	27	20	11	0

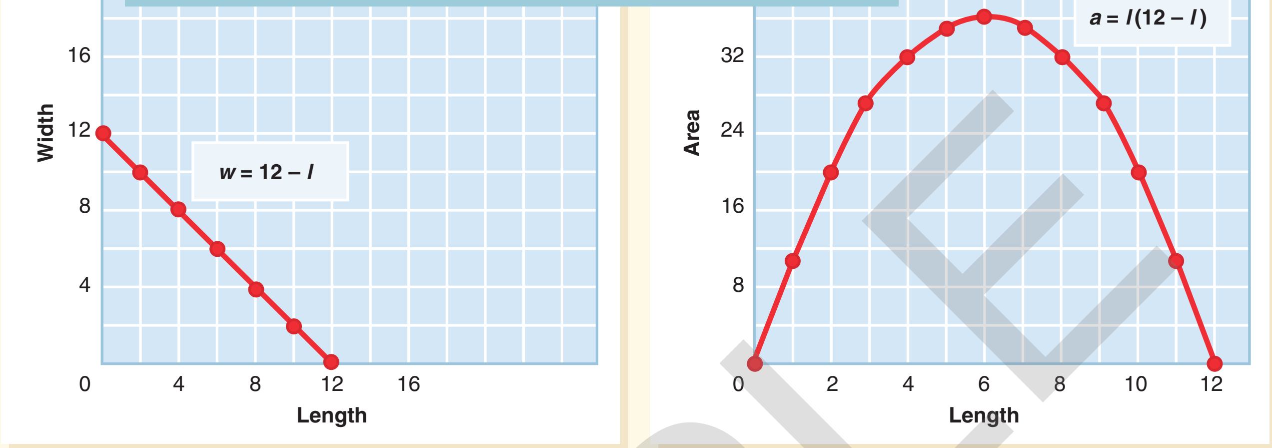


FIGURE 14.11 The width and area graphs as functions of the length of a rectangle with a fixed perimeter of 24 units.

a plant. Other rate contexts include hourly wages, gas mileage, profit, and cost of an item, such as a bus ticket.

Explorations of linear rate situations develop the concept of *slope*, which is the numeric value that describes the rate of change for a linear function. For example, one of the explicit formulas for the hexagon perimeter growing pattern is $y = 4x + 2$. Note that the rate of change is 4 because the perimeter increases by 4 with each new piece. All linear functions can be written in this form: $y = mx + b$ (including $y = mx$ when $b = 0$).

Conceptually, then, slope signifies how much y increases when x increases by 1. If a line contains the points (2,4) and (3,−5), you can see that as x increases by 1, y decreases by 9. So the rate of change, or slope, is −9. For the points (4,3) and (7,9), you can see that when x increases by 3, y increases by 6. Therefore, an increase of 1 in x results in a change of 2 in y (dividing 6 by 3). The slope is 2. After further exploration and experiences, your students will begin to generalize that you can find the rate of change or slope by finding the difference in the y values and dividing by the difference in the x values. Exploring this first through reasoning is important for students if they are to be able to make sense of and remember the formula for calculating slope when given two points. For an interactive tool that connects linear equations in the form $y = mx + b$ to graphs, try "Interactive Linear Equation" at Math Warehouse.

***Zero Slope*.** Understanding these two easily confused slopes requires contexts, such as walking rates. Consider this story:

You walk for 10 minutes at a rate of 1 mile per hour, stop for 3 minutes to watch a nest of baby birds, then walk for 5 more minutes at 2 miles per hour.

What will the graph look like for the 3 minutes when you stop? What is your rate when you stop? In fact, your rate is 0, and because you are at the same distance for 3 minutes, the graph will be a horizontal line.

No Slope. Let's say that you see a graph of a walking story that includes a vertical line—a line with no slope. What would this mean? That there is no change in the x variable—that you traveled a distance with no time passing! Now, even if you were a world record sprinter, this would be impossible. Remember that rate is based on a change of 1 in the x value.

Proportional and Nonproportional Situations. Linear functions can be proportional or nonproportional. A babysitter's pay is proportional to the hours he or she works (assuming an hourly rate). Proportional representations are shown in Figures 14.3(a) through (c). The hexagon pattern blocks, however, are nonproportional. Although you have a constant increase factor of 4, there are 2 extra units of perimeter (the sides of the two end blocks). Said another way, you cannot get from the input (number of blocks) to the perimeter by multiplying by a factor as you can in proportional situations.

All proportional situations, then, are equations in the form $y = mx$. Notice that the graphs of all proportional situations are straight lines that pass through the origin. Students will find that the slope of these lines is also the rate of change between the two variables.

In nonproportional situations, one value is constant. In the pattern blocks perimeter problem, for example, no matter which step number you are on, there are 2 units (one on each end) that must be added. Other examples include if you were walking at a constant rate but had a head start of 50 meters or if you were selling something and had an initial expense. The constant value, or initial value, can be found across representations beyond the contexts described here. In the table, it is the value when $x = 0$, which means it is the point where the graph crosses the y-axis. Context is important. The **Grocery Store** task, for example, asks students to figure out how long grocery carts are when they are pushed together. Students can see that it grows at a constant rate, but that there is a little extra on the end.

Nonproportional situations are more challenging for students to find the correspondence relationship or explicit rule. Students want to use the recursive value (e.g., +4) as the factor (×4). Also, in proportional situations it is true that there are twice as many in the twentieth term than in the tenth term, because the relationship is multiplicative only. But when there is a constant involved, this shortcut does not work, though students commonly make this error. Mathematics education researchers have found that having students analyze errors such as these is essential in helping support their learning of mathematics concepts (Lannin, Arbaugh, Barker, & Townsend, 2006).

Parallel, Same, and Perpendicular Lines. Students in eighth grade should be comparing different linear situations that result in parallel, same, or perpendicular lines (CCSSO, 2010). Using a context is necessary to build understanding.

Larry and Mary each earn $30 a day for the summer months. Mary starts the summer $50 dollars in debt, and Larry already has $20. In week 3, how much more money does Larry have? How much more does he have in week 7? When will Mary and Larry have the same amount of money?

The rates for Larry's and Mary's earnings are the same—and the graphs therefore go up at the same rate—that is, the slopes are the same. The graphs of Larry's earnings ($y = 30x + 20$) and Mary's earnings ($y = 30x - 50$) are parallel. We know this without even making the graphs because the rates (or slopes) are the same. Can you think of what change in Larry's and Mary's situations might result in the same line? Their initial value must be the same (and their rate).

Slopes can also tell us when two lines are perpendicular, but it is less obvious. A little bit of analysis with similar triangles will show that for perpendicular lines, the slope of one is the negative reciprocal of the other.

Complete Self-Check 14.3: Study of Patterns and Functions

Meaningful Use of Symbols

One reason students are unsuccessful in algebra is that they do not have a strong understanding of the symbols they are using. Symbols represent real situations and should be seen as useful tools for representing situations and solving real-life problems (e.g., calculating how many cookies we need to sell to make x dollars or at what rate do a given number of employees need to work to finish the project on time). Students cannot make sense of such situations without a strong understanding of mathematical symbols.

Looking at equivalent expressions that describe a context is an effective way to bring meaning to numbers and symbols. The classic task in Activity 14.10 involves such reasoning.

CCSS Standards for Mathematical Practice

MP4. Model with Mathematics.

Activity 14.10

CCSS-M: 6.EE.A.1; 6.EE.A.2a; b, c, 6.EE.A.3; 6.EE.A.4

Border Tiles

ENGLISH LANGUAGE LEARNERS

Ask students to build an 8×8 square array representing a swimming pool with colored tiles such that tiles of a different color are used around the border (Figure 14.12). Challenge students to find at least two ways to determine the number of border tiles used without counting them one by one. Students should use their tiles, words, and number sentences to show how they counted the squares. Ask students to illustrate their solution on centimeter grid paper. For ELLs, the drawing will be a useful support, but be sure the instructions are clear. There are at least five different methods of counting the border tiles around a square other than counting them one at a time.

A great online tool for this problem is the site titled "Plan Your Room." Input your dimensions (e.g., 8 feet 0 inches $\times$ 8 feet 0 inches) and click "Start with a Room."

Pause & Reflect

See whether you can find four or five different counting schemes for the "Border Tiles" problem. Can you see how the different expressions are equivalent? What questions might you pose to students in order to help them focus on these equivalent expressions?

Let's look at the various ways to count the border tiles. First, you may notice there are 10 squares across the top and across the bottom, leaving 8 squares on either side. This might be written as follows:

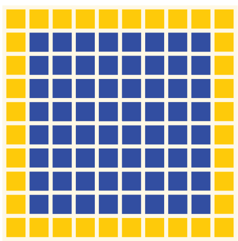


FIGURE 14.12 How many different ways can you find to count the border tiles of an 8×8 pool?

$$10 + 10 + 8 + 8 = 36 \text{ or } (2 \times 10) + (2 \times 8) = 36$$

Each of the following expressions can likewise be traced to looking at the squares in various groupings:

$$4 \times 9$$

$$4 \times 8 + 4$$

$$4 \times 10 - 4$$

$$100 - 64$$

CCSS Standards for Mathematical Practice

MP3. Construct viable arguments and critique the reasoning of others.

More equivalent expressions are possible because students may use addition instead of multiplication. Once the generalizations are created, ask students to justify how the elements in the expression map to the physical representation. Ask students to compare the different expressions and discuss whether they are all correct (and therefore equivalent) expressions for describing the general rule.

Notice that the task just completed involved numeric expressions—a good place to start. These expressions did not involve the two types of symbols that are perhaps the most important to understand—and, unfortunately, among the least well understood by many middle-school students. The equal sign ($=$) and inequality signs ($<$, $\leq$, $>$, $\geq$) are the first type. Variables are the second type. The sections that follow provide strategies for helping students understand these symbols.

Equal and Inequality Signs

The equal sign is one of the most important symbols in elementary arithmetic, in algebra, and in all mathematics. At the same time, research dating from 1975 to the present suggests that $=$ is a very poorly understood symbol (Kieran, 2007; RAND Mathematics Study Panel, 2003) and rarely represented in U.S. textbooks in a way to encourage students to understand the equivalence relationship—an understanding that is critical to understanding algebra (McNeil et al., 2006). The Common Core State Standards explicitly address developing an understanding of the equal sign as early as the first grade.

Why is it so important that students correctly understand the equal and inequality signs? First, it is important for students to understand and symbolize relationships in our number system. These signs are how we mathematically represent quantitative relationships. Conversely, when students fail to understand the equal sign, they typically have difficulty with algebraic expressions (Knuth, Stephens, McNeil, & Alibali, 2006). Consider the equation $5x + 24 = 54$. It requires students to see both sides of the equal sign as equivalent expressions. It is not possible to "do" the left-hand side. However, if both sides are understood as being equivalent, students will see that $5x$ must be 24 less than 54 or $5x = 30$. Therefore, x must equal 6.

Pause & Reflect

In the following expression, what number do you think belongs in the box?

$$8 + 4 = \square + 5$$

How do you think students in the early grades or in middle school typically answer this question?

In one study, no more than 10 percent of students from grades 1 to 6 put the correct number (7) in the box. The common responses were 12 and 17. (How did students get these answers?) In grade 6, not one student out of 145 put a 7 in the box (Falkner, Levi, & Carpenter, 1999).

Where do such misconceptions come from? A large majority of equations that students encounter in elementary school look like this: $5 + 7 =$ ___ or $8 \times 45 =$ ____. Naturally, students come to see $=$ as signifying "and the answer is" rather than a symbol to indicate equivalence (Carpenter et al., 2003; McNeil & Alibali, 2005; Molina & Ambrose, 2006).

Subtle shifts in the way you approach teaching computation can alleviate this major misconception. For example, rather than always asking students to solve a problem (like $45 + 61$ or 4×26), ask them to instead find an equivalent expression (Blanton, 2008). So, for $45 + 61$, students might write $45 + 61 = 40 + 66$. For a multiplication problem, students might write $4 \times 26 = 4 \times 25 + 4$ or $4 \times 26 = 2 \times (2 \times 26)$. Activity 14.11 is a way to work on equivalent expressions, while supporting the development of Making 10 strategy for learning the basic facts.

Activity 14.11

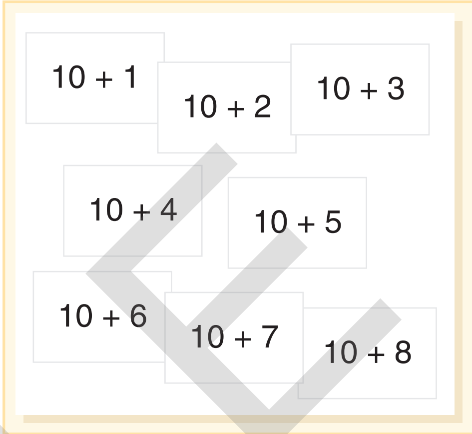
CCSS-M: 1.OA.C.6; 1.OA.D.7; 2.OA.B.2

Capture Ten

In this activity (based on Fosnot & Jacob, 2010), each pair of students will need eight note cards with the equations from 10 + 1 to 10 + 8 (see Equation Cards Activity Page). Students lay the cards out on their desks.

The partners will also need a deck of cards from which all face cards, aces, and ten cards have been removed. Together the partners each draw one playing card from the deck. They decide which note card is equivalent to the sum of their playing cards and place their cards behind the note card (e.g., if partners drew 8 and 5, they place their cards behind the 10 + 3 card). If the sum of the playing cards is less than 10, place the cards back in the deck. Note that the students do not need to actually add their cards; they just find an equivalent expression. Students can also play independently.

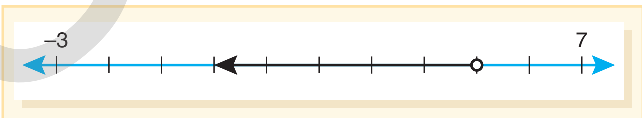
As an alternative to using note cards, you can prepare a game board whose spaces are all the equations from 10 + 1 to 10 + 8. The partners then place the sum of the playing cards drawn below the space on the game board.



Another way to support a stronger understanding of symbols is to encourage students to write their mental math strategies symbolically. For example, a student might explain that they solve $0.25 \times n$ by finding one-half of the number and then one-half again. In symbols, this might be written as: $0.25 \times 26 = \frac{1}{2}(\frac{1}{2} \times 26)$. This practice increases student understanding of the equal sign *and* the relationships among numbers and forms of numbers (e.g., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, so $0.5 \times 0.5 = 0.25$). Finally, as students are exploring the operations across the different situations, engage students in writing missing-addend and missing-factor equations, as well as equations with the result (answer) on the left (e.g., $50 = 5 \times 10$).

Inequalities are also poorly understood and have not received less attention than the equal sign, likely because inequalities are not as prevalent in the curriculum or in real life. In the CCSS-M standards, they are first mentioned in grade 6. Understanding and using inequalities is important and can be introduced earlier as a way for students to compare quantities: Are these two quantities equal or is one greater than the other?

The number line is a valuable tool for understanding inequalities. For example, students can be asked to show $x < 5$ on the number line:



A context such as money can provide a good way to make sense of inequalities, as in the following example.

You have $100 for purchasing gift cards for your 5 friends. You want to spend the same on each, and you will also need to spend $10 to buy a package of card-holders for the gift cards. Describe this situation with symbols.

Pause & Reflect

How would you write this inequality? How might students write it? What difficulties do you anticipate? And, importantly, what questions will you pose to help students build meaning for the inequality symbols?

Students might record the situation in any of the following ways (using *a* for the amount of money for the gift):

$$5a + 10 \leq 100 \qquad 10 + 5a \leq 100 \qquad 100 \geq 10 + 5a \qquad 100 \geq 5a + 10$$

They may also make these inequalities without the equal signs: $<$ and $>$. Discuss with students what it means to say "less than" or "less than or equal to." Invite students to debate which signs make more sense given the situation. Graph the result and see if the graph makes sense given the situation.

Deciding whether to use the less than or greater than sign can be confusing for students. Invite students to say in words what the inequality means. For example, the first statement directly translates to 5 gift cards and $10 for a package of holders must be less than or equal to $100. The final example directly translates to I have $100, which must be more than or the same as the cost of 5 cards and the holders. Ask questions that help students analyze the situation quantitatively, such as, "Which has to be more, the amount you have or the amount you spend?"

***FORMATIVE ASSESSMENT* Notes.** Ask students to **write** a real-life story problem that involves an inequality. You can add expectations such as "It must be multi-step" and "you must illustrate the solution on a number line" (for more details see Whaley, 2012, for a full lesson, examples, rubric and discussion). Writing helps students connect representations and helps you see what misconceptions they might have.

Conceptualizing the Equal Sign as a Balance. Helping students understand the idea of equivalence can and must be developed concretely. The next two activities illustrate how kinesthetic approaches, tactile objects, and visualizations can reinforce the "balancing" notion of the equal sign.

Activity 14.12

CCSS-M: 1.OA.D.7; 2.NBT.A.4

Seesaw Students

STUDENTS *with* SPECIAL NEEDS

Ask students to raise their arms to look like a seesaw. Explain that you have big juicy oranges, all weighing the same, and tiny little apples, all weighing the same. Ask students to imagine that you have placed an orange in each of their left hands (students should bend to lower left side). Ask students to imagine that you place another orange on the right side (students level off). Next, with oranges still there, ask students to imagine an apple added to the left. Finally, say you are adding another apple, which is going on the left (again). Then ask them to imagine the apple moving over to the right. This is a particularly important activity for students with disabilities, who may be challenged with the abstract idea of balancing values of expressions.

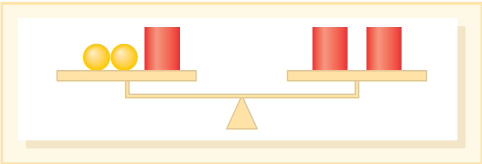
After acting out several seesaw examples, ask students to write Seesaw Findings (e.g., "If you have a balanced seesaw and add something to one side, it will tilt to that side," and "If you take away the same object from both sides of the seesaw, it will still be balanced").

Activity 14.13

CCSS-M: 6.EE.A.2a; 6.EE.B.5; 6.EE.B.6; 7.EE.B.4a, b

What Do You Know about the Shapes?

Present a scale with objects on both sides. Here is an example:



Ask students what they know about the shapes: "The red cylinders weigh the same. The yellow spheres (balls) weigh the same. What do you know about how the weights of the spheres and the cylinders compare?" Figure 14.13 illustrates how a third grader explained what she knew. (Notice that these tasks, appropriate for early grades, are good beginnings for the more advanced balancing tasks later in this chapter.)

For more explorations like this, see "Pan Balance—Shapes" on NCTM's Illuminations website.

After students have experiences with these shapes, they can then explore numbers, eventually moving on to variables. Figure 14.14 offers examples that connect the balance to the related equation. This two-pan balance model also illustrates that the expressions on each side represent a number.

Activity 14.14

CCSS-M: 2.NBT.A.4; 4.NBT.A.1; 5.NBT.A.3a, b; 6.EE.A.4

Tilt or Balance

STUDENTS *with* SPECIAL NEEDS

Draw or project a simple two-pan balance. In each pan, write a numeric expression and ask which pan will go down or whether the two will balance (see Figure 14.14(a)). Select expressions appropriate for the grade level of students (e.g., sums within 100 for grade 2 and sums of fractions for grade 5). Challenge students to write expressions for each side of the scale to make it balance. For each, write a corresponding equation to illustrate the meaning of $=$. Note that when the scale "tilts," either a "greater-than" or "less-than" symbol ($>$ or $<$) is used, and if it is balanced, an equal sign ($=$) is used. Include examples (like the third and fourth balances in the figure) for which students can make the determination by analyzing the relationships on the two sides rather than by doing the computation. For students with disabilities, instead of having them write expressions for each side of the scale, share a small collection of cards with expressions and have them identify the ones that will make the scale balance.

As an alternative or extension, use missing-value expressions. Ask students to find a number that will result in one side tilting downward, a number that will result in the other side tilting downward, and one that will result in the two sides being balanced (see Figure 14.14(b)). (See Expanded Lesson Tilt or Balance for more details on teaching this activity.)

The balance is a concrete tool that can help students understand that if you add or subtract a value from one side, you must add or subtract a like value from the other side to keep the equation balanced.

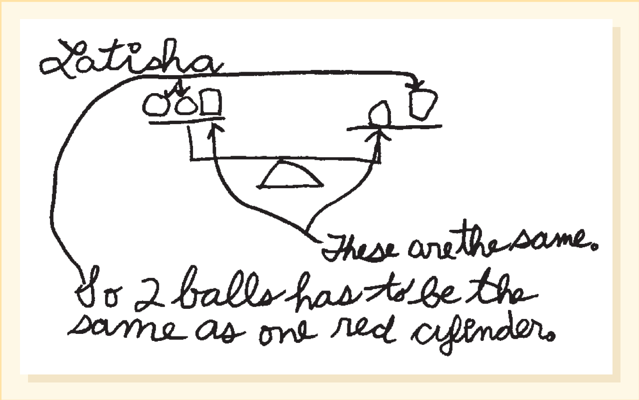


FIGURE 14.13 Latisha's work on the problem.

Source: Figure 4 from Mann, R. L. (2004). "Balancing Act: The Truth Behind the Equals Sign." *Teaching Children Mathematics, 11*(2), p. 68. Reprinted with permission. Copyright © 2004 by the National Council of Teachers of Mathematics. All rights reserved.

There are several excellent activities online:

- PBS Cyberchase Poddle Weigh-in: Shapes are balanced with numbers between 1 and 4.
- Agame Monkey Math Balance: Students select numbers for each side of a balance to make the two sides balance (level of difficulty can be adapted).
- NCTM Illuminations Pan Balance—Numbers or Pan Balance—Expressions provide a virtual balance where students can enter what they believe to be equivalent expressions (with numbers or symbols).

Figure 14.15 shows solutions for two equations, one in a balance and the other without. Even after you have stopped using the balance, it is a good idea to refer to the two-pan balance concept of equality and the idea of keeping the sides balanced. This use of concrete (an actual balance) or semi-concrete (drawings of a balance) representations helps students develop a strong understanding of the abstract concept of equality.

The notion of preserving balance also applies to inequalities—but what is preserved is imbalance. In other words, if one side is more than the other side and you subtract 5 from both, the one side is still more.

In middle school, students begin to manipulate equations so that they are easier to graph and/or to compare to other equations. These experiences help ground a student's understanding of how to preserve equivalence when moving numbers or variables across the equal sign.

True/False and Open Sentences. Carpenter and colleagues (2003) suggest that a good starting point for helping students with the equal sign is to explore equations as either true or false. Clarifying the meaning of the equal sign is just one of the outcomes of this type of exploration, as seen in the following activity.

Activity 14.15

CCSS-M: 1.OA.B.3; 1.OA.D.7; 1.NBT.B.4; 2.NBT.B.5; 3.OA.B.5; 4.NBT.B.5; 5.NF.A.1

True or False?

Introduce true/false sentences or equations with simple examples to explain what is meant by a true equation and a false equation. Then put several simple equations on the board, some true and some false. The following are appropriate for primary grades:

$7 = 5 + 2 \qquad 4 + 1 = 6$

$4 + 5 = 8 + 1 \qquad 8 = 10 - 1$

Your collection might include other operations, but keep the computations simple. Ask students to talk to their partners and decide which of the equations are true equations (and why) and which are not (and why not).

For older students, use fractions, decimals, and larger numbers.

$120 = 60 \times 2 \qquad 318 = 318$

$\frac{1}{2} = \frac{1}{4} + \frac{1}{4} \qquad 1 = \frac{3}{4} + \frac{2}{1} \qquad 345 + 71 = 70 + 344$

$1210 - 35 = 1310 - 45 \qquad 0.4 \times 15 = 0.2 \times 30$

Listen to the types of reasons that students use to justify their answers, and plan additional equations accordingly. ELLs and students with disabilities will benefit from first explaining (or showing) their thinking to a partner (a low-risk speaking opportunity) and then sharing with the whole group. "Pan Balance—Numbers" on NCTM's Illuminations website can be used to explore and/or verify equivalence.

ENGLISH LANGUAGE LEARNERS

STUDENTS *with* SPECIAL NEEDS

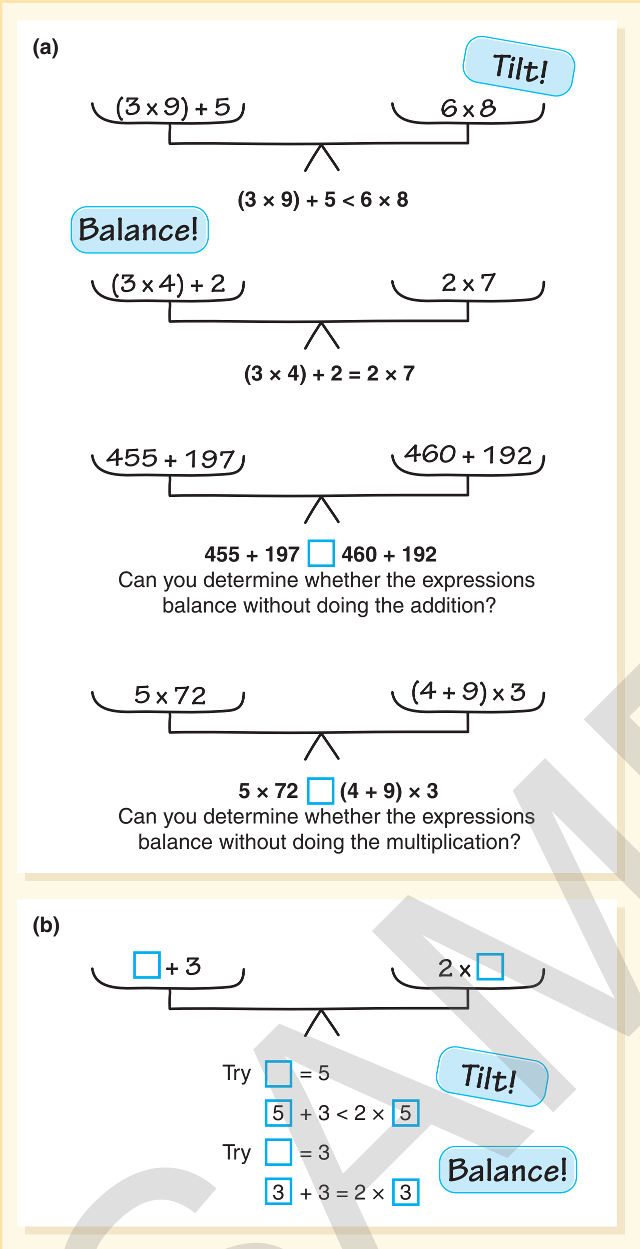


FIGURE 14.14 Using expressions and variables in equations and inequalities.

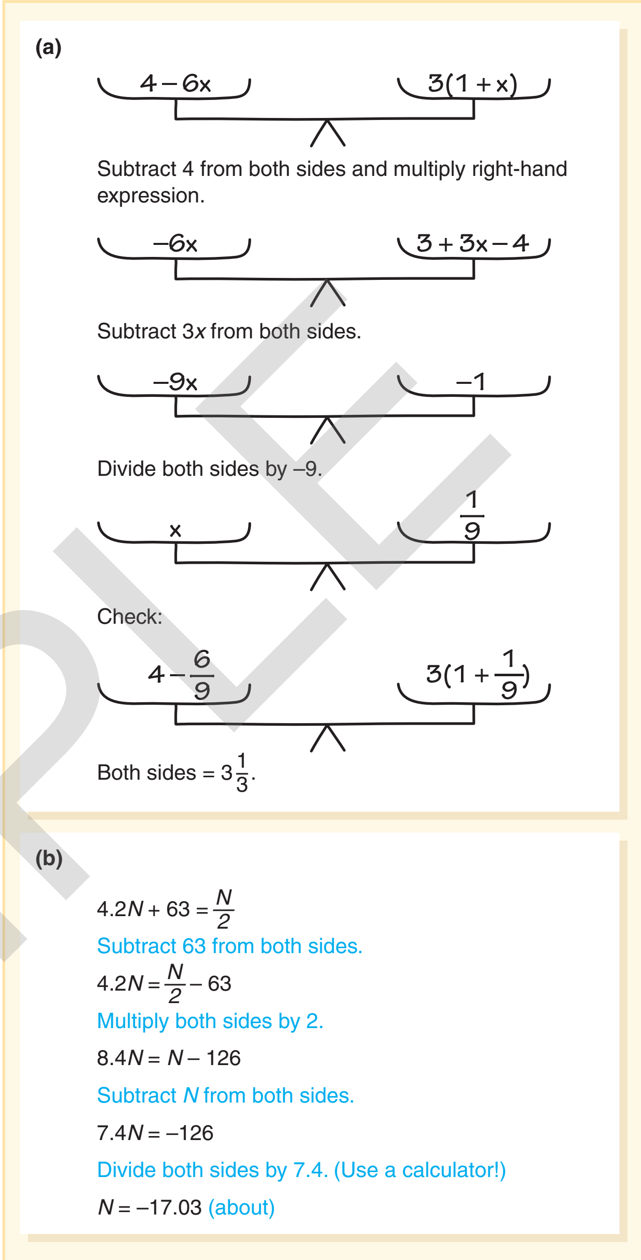


FIGURE 14.15 Using a balance scale to think about solving equations.

Students will generally agree on equations when there is an expression on one side and a single number on the other, although initially the less familiar form of $7 = 5 + 2$ may generate discussion. For an equation with no operation ($8 = 8$), the discussion may be lively. Reinforce that the equal sign means "**is the same as**" by using that language when you read the symbol. Inequalities should be explored in a similar manner.

After students have experienced true/false sentences, introduce an open sentence—one with a box to be filled in or a variable. As early as first grade students can understand and benefit from using variables (Blanton et al., 2011). To develop an understanding of open sentences, encourage students to look at the number sentence holistically and discuss in words what the equation represents.

Activity 14.16

CCSS-M: 1.OA.D.7; 1.OA.D.8; 2.OA.A.1; 3.OA.A.4; 5.OA.A.2; 6.EE.A.3; 6.EE.B.5

What's Missing?

Prepare a set of missing-value equations, using numbers appropriate for the grade level. A sampling across the grades is provided below. Ask students to figure out what is missing and how they know. Notice that the equations are set up so that students do not always have perform the operations to figure out what is missing. Encourage them to look at the equation and see if they can figure out what is missing without solving it. Probe to see if there is more than one way to find what is missing.

Here is a sampling of ideas across the grades:

$4 + \square = 6$	$4 + 5 = \square - 1$	$\square + 5 = 5 + 8$	$3 \times 7 = 7 \times \square$
$15 + 27 = n + 28$	$12 \times n = 24 \times 5$	$6 \times n = 3 \times 8$	$15 \times 27 = n \times 27 + 5 \times 27$
$0.5 + a = 5$	$4.5 + 5.5 = a + 1$	$a \times 4 = 4.8$	$2.4 \div a = 4.8 \div 6$
$3.6 - n = 3.7 - 4$	$n + 0.5 = 0.5 + 4.8$	$15 \times 27 = n \times 27 + 5 \times 27$	$1 = 0.5 \div n$

Relational Thinking. Students may think about equations in three ways, each developmental in nature (Stephens et al., 2013). First, as noted previously, they may have an *operational view*, meaning that the equal sign means "do something." Second, students develop a *relational–computational* view. At this phase, students understand that the equal sign symbolizes a relation between answers to two calculations, but they only see computation as they way to determine if the two sides are equal or not. Finally, students develop a *relational–structural view* of the equal sign (we will refer to this as relational). In this thinking, a student uses numeric relationships between the two sides of the equal sign rather than actually computing the amounts.

Consider two distinctly different explanations for placing an 8 in the box for the open sentence $7 + n = 6 + 9$.

a. Since $6 + 9$ is 15, I need to figure out 7 plus what equals 15. It is 8, so n equals 8.
b. Seven is one more than the 6 on the other side. That means that n should be one less than 9, so it must be 8.

The first student computes the result on one side and adjusts the result on the other to make the sentence true (relational–*computational* approach). The second student uses a relationship between the expressions on either side of the equal sign. This student does not need to compute the values on each side (relational–*structural* approach). When the numbers are large, a relational-structural approach is much more efficient and useful.

Pause & Reflect

How are the two students' correct responses for $7 + n = 6 + 9$ different? How would each of these students solve this open sentence?

$$534 + 175 = 174 + \_\_\_\_$$

The first student will do the computation and will perhaps have difficulty finding the correct addend. The second student will reason that 174 is one less than 175, so the number in the box must be one more than 534.

***FORMATIVE ASSESSMENT* Notes.** As students work on these types of tasks, you can **interview** them one on one (though you may not get to everyone). Listen for whether they are using relational-structural thinking. If they are not, ask, "Can you find the answer without actually doing any computation?" This questioning helps nudge students toward relational thinking and helps you decide what instructional steps are next. ■

Students need many and ongoing opportunities to explore problems that encourage relational thinking (Stephens et al., 2013). Explore increasingly complex true/false and open sentences with your class, perhaps as daily problems, warm-ups, enrichment, or at stations. Use large numbers that make computation difficult (not impossible) and multistep equations as a means to encourage relational-structural thinking. Here are some examples:

True/False

$674 - 389 = 664 - 379$ $\qquad$ $42 = 0.5 \times 84$

$\frac{2}{5} = \frac{1}{2} + \frac{1}{3}$ $\qquad$ $64 \div 14 = 32 \div 28$

Open Sentences

$126 - 37 = n - 40$ $\qquad$ $7.03 + 0.056 = 7.01 + n$

$20 \times 48 = n \times 24$ $\qquad$ $4800 \div 25 = n \times 48$

Multistep Sentences

$512 \times 5 \times 20 = n$ $\qquad$ $68 + 58 = 57 + 69 + n$

$\frac{3}{10} + n + \frac{1}{10} = \frac{2}{5} + \frac{1}{5}$ $\qquad$ $37 \times 18 \div 37 = n$

Pause & Reflect

Look at each of the multistep sentences and think about the order in which you can solve it. Can relational-structural thinking help make the problem easier to solve? ●

Notice that the careful construction of these equations encourages relational thinking. Each of the mulistep problems can be solved mentally if the first step of solving is chosen carefully. Selecting equations such as this encourage students to look at equations in their entirety rather than just jumping right into a series of computations, an important aspect of algebraic thinking (Blanton et al., 2011).

Molina and Ambrose (2006), researchers in mathematics education, used the true/false and open-ended prompts with third graders who did not have a relational understanding of the equal sign. For example, all 13 students answered $8 + 4 = \_\_\_ + 5$ with 12. They found that asking students to write their own open sentences was particularly effective in helping students solidify their understanding of the equal sign (see Activity 14.17).

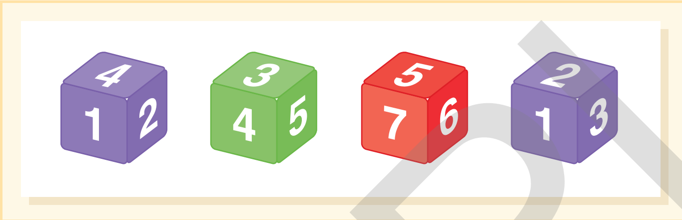
Activity 14.17

CCSS-M: 1.OA.D.7; 1.OA.D.8; 2.OA.A.1; 3.OA.A.4; 4.NF.B.3a; 5.OA.A.2

Make a Statement!

STUDENTS *with* SPECIAL NEEDS

Ask students to write their own true/false and open sentences that they can use to challenge their classmates. This works for both equations and inequalities! To support student thinking, provide dice with numerals on them. They can turn the dice to different faces to try different possibilities.



Ask students to write three equations (or inequalities) with at least one true and at least one false sentence. For students who need additional structure, in particular students with disabilities, consider providing frames such as these:

___ + ___ = ___ + ___	___ + ___ > ___ + ___
___ − ___ = ___ − ___	___ + ___ < ___ − ___
___ + ___ = ___ − ___	___ + ___ ≥ ___ − ___

(or use multiplication and division)

Students can trade their set of statements with other students to find the False Statement. Interesting equations/inequalities can be the focus of a follow-up full-class discussion.

When students write their own true/false sentences, they often are intrigued with the idea of using large numbers and lots of numbers in their sentences. This encourages them to create sentences involving relational-structural thinking.

The Meaning of Variables

Variables can be interpreted in many ways. Variables are first mentioned in the CCSS-M standards in grade 6, but researchers suggest starting much earlier so that students are more adept at using variables when they encounter more complex mathematical situations in middle school (Blanton et al., 2011). Variables can be used to represent a unique but unknown quantity or represent a quantity that varies. Unfortunately, students often think of the former (the variable is a placeholder for one exact number) and not the latter (the a variable could represent multiple, even infinite values). As discussed in functions, variables are also used to describe a pattern. Experiences in elementary and middle school should focus on building meaning for both, as delineated in the next two sections.

Variables Used as Unknown Values. In the open sentence explorations, the □ is a precursor of a variable used in this way. Even in the primary grades, you can use letters instead of a box in your open sentences, such as an n standing for the missing number.

Many story problems involve a situation in which the variable is a specific unknown, as in the following basic example:

Gary ate 5 strawberries and Jeremy ate some, too. The container of 12 was gone! How many did Jeremy eat?

Although students can solve this problem without using algebra, they can begin to learn about variables by expressing it in symbols: $5 + s = 12$. These problems can grow in difficulty over time.

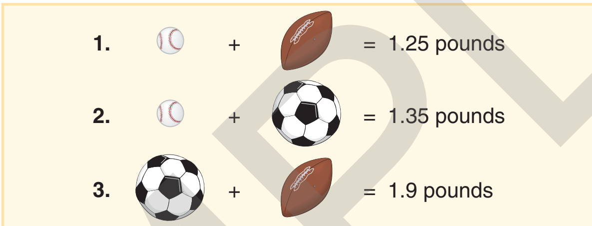
With a context, students can even explore three variables, each one standing for an unknown value, as in the following activity (adapted from Maida, 2004).

Activity 14.18

CCSS-M: 6.EE.A.4; 7.EE.A.2; 8.EE.C.8b

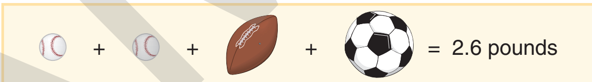
Ball Weights

Students will figure out the weight of three balls, given the following three facts:



Ask students to look at each fact and make observations that help them generate other facts. For example, they might notice that the soccer ball weighs 0.1 pound more than the football. Write this in the same fashion as the other statements. Continue until these discoveries lead to finding the weight of each ball. Encourage students to use models to represent and explore the problem.

One possible approach: Add equations 1 and 2:



Then take away the football and soccer ball, reducing the weight by 1.9 pounds (based on the information in equation 3), and you have two baseballs that weigh 0.7 pound. Divide by 2, so one baseball is 0.35 pounds.

You may recognize this last example as a system of equations presented in a visual. In this format, it can be solved by reasoning, making it accessible to upper elementary and middle school students.

Another concrete way to work on systems of equations is through balancing. Notice the work done in building the concept of the equal sign is now applied to understanding and solving for variables.

In Figure 14.16, a series of examples shows problems in which each shape on the scales represents a different value. Two or more scales for a single problem provide different information about the shapes or variables. Problems of this type can be adjusted in difficulty for students across the grades. The NLVM applet Algebra Balance Scales and Algebra Balance Scales—Negative is an excellent tool for learning about balancing equations.

When no numbers are involved, as in the top two examples of Figure 14.16, students can find combinations of numbers for the shapes that make all of the balances balance. If an arbitrary value is given to one of the shapes, then values for the other shapes can be found accordingly.

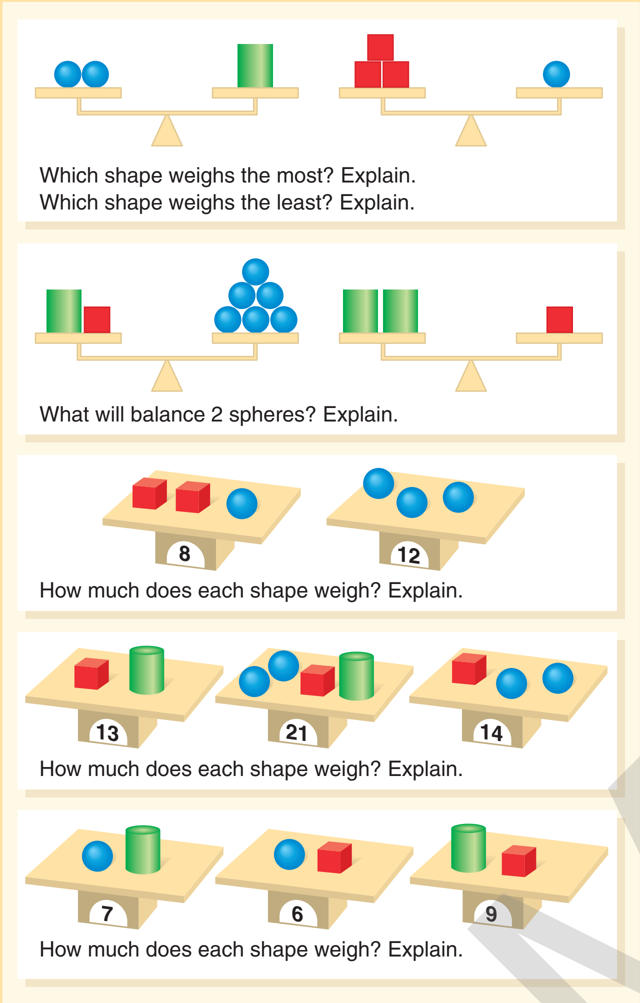


FIGURE 14.16 Examples of problems with multiple variables and multiple scales.

In the second example, if the sphere equals 2, then the cylinder must be 4 and the cube equals 8. If a different value is given to the sphere, the other shapes will change accordingly.

Pause & Reflect

How would you solve the last problem in Figure 14.16? Can you solve it in two ways?

You (and your students) can tell whether you are correct by checking your solutions against the original scale positions. Believe it or not, you have just solved a system of equations, a skill generally found in a formal algebra class.

Simplifying Expressions and Equations. Simplifying equations and solving for x have often been meaningless tasks, and students are unsure of why they need to know what x is or what steps to do and in what order. This must be taught in a more meaningful way! Knowing how to simplify and recognizing equivalent expressions are essential skills for working algebraically. Students are often confused about what the instruction "simplify" means. (Imagine an ELL wondering why the teacher is asking students to change the original problem to an easier one.) The Border Tiles problem provides a good context for thinking about simplifying and equivalence. Recall that there are at least 5 possibilities for finding the number of border tiles. If the pool had dimensions other than 8×8, those equations would be structurally the same, but with different values. If the square had a side of length p, the total number of tiles could be found in similar ways:

$10 + 10 + 8 + 8$	$(2 \times 10) + (2 \times 8)$	4×9	$100 - 64$
$(p+2)+(p+2)+p+p$	$2 \times (p + 2) + (2 \times p)$	$4 \times (p + 1)$	$(p + 2)^2 - p^2$

Invite students to enter these expressions into the Table function on their graphing calculator and graph them to see whether they are equivalent (Brown & Mehilos, 2010). Looking at these options, the connection can be made for which one is stated the most simply (briefest or easiest to understand).

Students need an understanding of how to apply mathematical properties and how to preserve equivalence as they simplify. (This is one of the Common Core State Standards in grade 7.) In addition to the ideas that have been offered (open sentences, true/false sentences, etc.), one way to do this is to have students look at simplifications that have errors and explain how to fix the errors (Hawes, 2007). Figure 14.17 shows how three students have corrected the simplification of $(2x + 1) - (x + 6)$. You can create your own examples of simplified expressions that have an error in them—select an error that is commonly made in your classroom so that the class can discuss why the expression is not correct.

Activity 14.19 provides an engaging way for students to explore properties and equivalent expressions.

Activity 14.19

CCSS-M: 5.OA.A.2; 6.EE.A.2a; 7.EE.A.2

Solving the Mystery

STUDENTS *with* SPECIAL NEEDS

Begin by having students do the following sequence of operations:

Write down any number.

Add to it the number that comes after it.

Add 9.

Divide by 2.

Subtract the number you began with.

Now, you can "read their minds." Everyone ended up with 5! Ask students, "How does this trick work? [Start with n. Add the next number: $n + (n + 1) = 2n + 1$. Adding 9 gives $2n + 10$. Dividing by 2 leaves $n + 5$. Now, subtract the number you began with, leaving 5.] See also a second Solving the Mystery task. For students with disabilities or students who struggle with variables, suggest that instead of using an actual number they use an object, such as a cube, and physically build the steps of the problem, as illustrated in Figure 14.18. In this Mystery, the result is a two-digit number where the tens place is the first number selected and the second digit is the second number selected (ask students to explain how this happened). As a follow-up or for enrichment, students can generate their own number tricks.

Explain how to fix this simplification. Give reasons.
$(2x + 1) - (x + 6) = 2x + 1 - x + 6$

Gabrielle's solution

If $x = 3$ then the order of operations would take place, so the problem would look like $(2 \cdot 3 + 1) - (3 + 6) = 2 \cdot 3 + 1 - 3 + 6$ you would have to do $1 - 3$ instead of $1 + 6$. But its actually $3 + 6$. so that's the mistake.

Prabdheep's solution

The problem will look like this in its correct form $(2x + 1) - (x + 6) = 2x + {}^-1x + {}^-6$ because there is a minus sign right outside of the () On the left side it means its -1. So if you times -1 by x its $-1x$ not 1-x. When you times -1 by 6 its $^-6$ not 6.

Briannon's solution

Explain how to fix this problem. Give Reasons
$(2x + 1) - (x + 6) = 2x + 1 - x + 6$
you are subtracting x and 6 not subtracting x and adding 6
correctly simplified the problem is
$(2x + 1) + {}^-(x + 6)$ distribute negative
$2x + 1 + {}^-x + {}^-6$
$\boxed{x + {}^-5}$

FIGURE 14.17 Three students provide different explanations for fixing the flawed simplification.

Source: Figure 3 from Hawes, K. (2007). "Using Error Analysis to Teach Equation Solving." *Mathematics Teaching in the Middle School, 12*(5), p. 241. Reprinted with permission. Copyright © 2007 by the National Council of Teachers of Mathematics. All rights reserved.

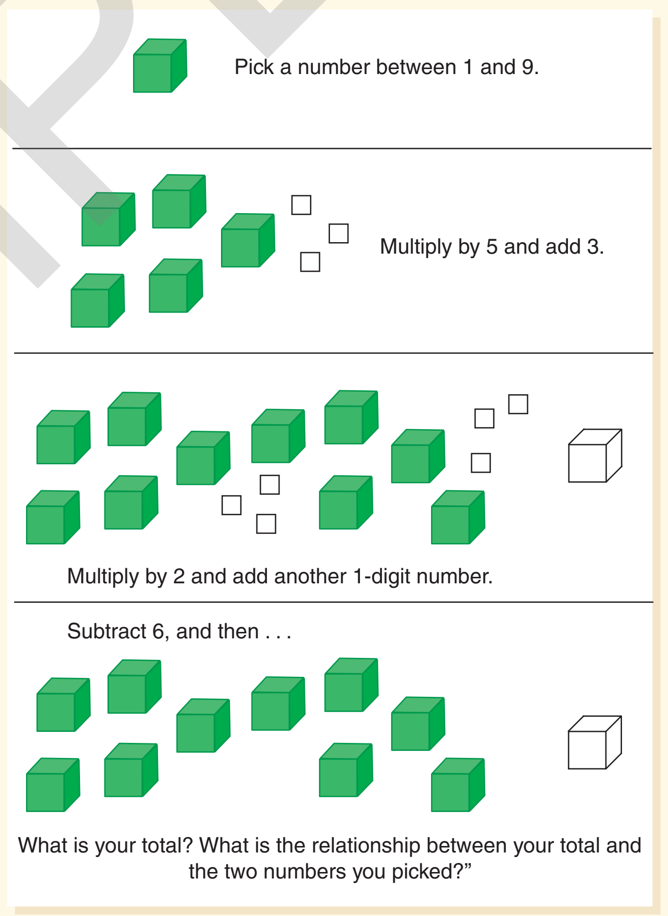


FIGURE 14.18 Cubes can illustrate the steps in "Solving the Mystery."

Solving systems of equations has also been presented in a way that includes a series of procedures with little attention to meaning (e.g., by graphing, by substitution, and simultaneously). Mathematically proficient students should have access to multiple approaches, including these three (CCSSO, 2010). However, rather than learn one way by rote each day or be tested on whether they can use each approach, they should encounter a system and be guided by the question when you ask, "How can we determine the point of intersection of this system?" Just as they did with the operations, students should *choose* a method for solving a system of equations that fits the situation, using appropriate tools.

Graphing calculators make the choice of using a graph to determine the point of intersection an efficient option, whereas graphing by hand used to be one of the most tedious methods. Also among the strategies that students must try is observation. Too often, students leap into solving a system algebraically without stopping to observe the values in the two equations. Look at the systems of equations here, and see which ones might be solved for x or y by observation.

$x + y = 25$ and $x + 2y = 25$

$3x + y = 20$ and $x + 2y = 10$

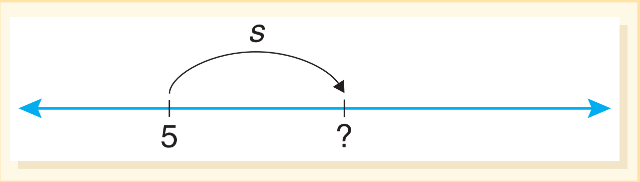
$8x + 6y = 82$ and $4x + 3y = 41$

$\frac{y}{3} = 5$ and $y + 5x = 60$

Variables Used as Quantities That Vary. As noted earlier, the important concept that variables can represent more than one missing value is not well understood by students and is not as explicit in the curriculum as it should be. Students need experiences with variables that vary early in the elementary curriculum. Young students can begin to describe patterns using variables. For example, when describing the how many legs for any number of dogs, students might write $L = 4 \times D$, meaning that the number of legs is four times the number of dogs. Importantly, you must emphasize that the variable stands for *the number of* because students can confuse the variable to be a label (Blanton et al., 2011). Let's revisit the previous story above, but remove the result:

Gary ate 5 strawberries and Jeremy ate some, too. How might you describe the total number of strawberries eaten?

Because the total has been removed, the goal becomes writing the expression $(5 + s)$. This can also be illustrated on a number line:



The number line is an important model in developing the concept of variable. As illustrated in Figure 14.19, finding where variables are in relation to numbers and in relation to other variables helps to build meaning (Darley, 2009).

When students are looking at the number line, ask questions like, "What is the value of x? Can it be any number? If we don't know what x is, how can we place $\frac{1}{3}x$ on the number line?" "Think of a value that x cannot be." Notice that in the two examples, x really can be any positive value. However, if you place $x + 2$ on the number line somewhere close to x, the space between these is 2, and you can use this distance as a "measure" to approximate the size of x. Because students use the number line extensively with whole numbers, it is a good way to bridge to algebra. Having an algebra number line posted in your room where you can trade out what values are posted can provide many opportunities to think about the relative value of variables.

Context is important in writing equations with variables. Compare the two problems here (Blanton, 2008):

1. Annie has \$10. Noah has \$3 more than Annie. How much money does Noah have?
2. Annie has some money. Noah has \$3 more than Annie. How much money does Noah have?

Primary grade students can list possible ways in a table and eventually represent the answer as *Annie* + *3* = *Noah* or more briefly, $A + 3 = N$.

The following example is appropriate for middle school students as a context for exploring variables that vary:

If you have \$10 to spend on \$2 granola bars and \$1 fruit rolls, how many ways can you spend all your money without receiving change?

To begin exploring this problem, students record data in a table and look for patterns. They notice that when the number of granola bars changes by 1, the number of fruit rolls changes by 2. Symbolically, this representation is $2g + f = \$10$, where g is the number of granola bars and f is the number of fruit rolls.

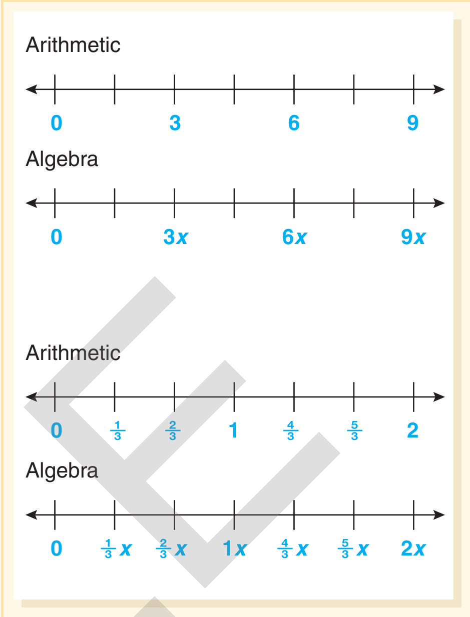


FIGURE 14.19 Using the number line to build meaning of variables.

It is also important to include decimal and fraction values in the exploration of variables. As any algebra teacher will confirm, students struggle most with these numbers—again resulting from the lack of earlier, more concrete, and visual experiences mixing fractions and decimals with variables. For example, if you were buying \$1.75 pencils and \$1.25 erasers from the school store and spent all of \$35.00, how many combinations are possible? What equation represents this situation?

For students with special needs or students who might be unfamiliar with using a table, it is helpful to adapt the table to include both how many and how much, as shown in Figure 14.20. Reinforce the two elements with each entry (how many and how much). In addition, calculators can facilitate exploration of possible solutions. To increase the challenge for advanced or gifted students, ask students to graph the values or to consider more complex situations.

Once students have the expression in symbols (in this case, $1.75x + 1.25y = 35.00$), ask students to tie each number and variable back to the context. In this way, students can make sense of what is normally poorly understood and really develop a strong foundation for the algebra they will study in secondary school.

			Total \$35.00
\$1.75 item		**\$1.25 item**	
	\$35.00		\$0
20		**0**	
	\$0		\$35
0		**28**	

FIGURE 14.20 A table adapted to include how many and how much for each row.

Source: Hyde, A., George, K., Mynard, S., Hull, C., Watson, S., and Watson, P. (2006). "Creating Multiple Representations in Algebra: All Chocolate, No Change," *Mathematics Teaching in the Middle School, 11*(6), 262–268. Reprinted with permission. Copyright © 2006 by the National Council of Teachers of Mathematics. All rights reserved.

Independent and Dependent Variables. Although the meanings of *independent variables* and *dependent variables* are implied by the words themselves, the concepts can still be challenging for students. The independent variable is the step number, or the input, or whatever value is being used to find another value. For example, in the case of the strings of pattern blocks, the independent variable is the number of blocks in the string. The dependent variable is the number of objects needed, the output, or whatever value you get from using the independent variable. In the pattern blocks problem, it is the perimeter. You can say that the perimeter of the block structure depends on the number of blocks. Recall the two equations and graphs representing a pen of 24 meters in Figure 14.11. In this case, the length has been selected as the independent variable (though it could have as easily been the width), and the dependent variable is width. What are the independent and dependent variables for the Border Tiles (Figure 14.12)?

Complete Self-Check 14.4: Meaningful Use of Symbols

Mathematical Modeling

Modeling with mathematics is one of the eight Standards for Mathematical Practice in the Common Core State Standards.

> Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. (CCSSO, 2010, p. 72)

Mathematical models are not to be confused with visual models such as manipulatives or drawings for building a pattern (such as pattern blocks or centimeter grid paper).

We have already seen many examples of mathematical models (e.g., the model or equation for describing the number of tiles required for a pool of various dimensions). The equation, or mathematical model, allows us to find values that cannot be observed in the real phenomenon.

Activity 14.20 provides another context appropriate for developing a mathematical model.

Activity 14.20

CCSS-M: 7.EE.A.2; 7.EE.B.4a; 8.F.B.4

How Many Gallons Left?

ENGLISH LANGUAGE LEARNERS

Ask students to create equations to describe the gallons left for given miles traveled (assuming that you started the trip with a full tank of gasoline).

For example:

A car gets 27 miles per gallon of gas. It has a gas tank that holds 15 gallons.

A van gets 18 miles per gallon and has a tank that holds 20 gallons.

The mathematical model or equation in the first situation is $g = 15 - \frac{m}{27}$. After students write an equation, use it to answer questions about the trip. For example, "How can you tell from the model how much gas will be left after driving 300 miles?" "How many miles can you drive before the gas tank has only 3 gallons left?" ELLs may be more familiar with kilometers per liter, which means you can adapt the problem to those units or connect the meaning of the two.

1. Pleasant's Hardware buys widgets for \$4.17 each, marks them up 35 percent over wholesale, and sells them at that price. Create a mathematical model to relate widgets sold (***w***) to profit (***p***). The manager asks you to determine the formula if she were to put the widgets on sale for 25 percent off. What is your formula or mathematical model for the sale, comparing widgets sold (***s***) to profit (***p***)?
2. In Arches National Park in Moab, Utah, there are sandstone cliffs. A green coating of color, called cyanobacteria, covers some of the sandstone. The cyanobacteria grow by splitting into two (or doubling) in a certain time period. If the sandstone started with 50 bacteria, create a mathematical model for describing the growth of cyanobacteria on the sandstone. (See Buerman, 2007, for more on exploring Arches National Park mathematically.)

FIGURE 14.21 Mathematical modeling problems for further exploration.

Creating equations to describe situations is a very important skill and students need multiple opportunities to translate situations into equations where the goal is the mathematical model (and not a solution). Two more engaging contexts are provided in Figure 14.21.

Sometimes a model is provided, and the important task is for students to understand and use the formula. Consider the following classic pumping water problem based on the Michigan Algebra Project task (Herbel-Eisenmann & Phillips, 2005):

You turn on a pump to empty the water from a swimming pool. The amount of water in the pool (W, measured in gallons) at any time (T, measured in hours) is given by the following equation: $W = -350\,(T-4)$.

Pause & Reflect

What questions might you pose to middle school students to help them make sense of this equation? Try to think of three.

In the Michigan Algebra Project, students were asked to solve several problems and explain how the equation was used to find the answer. Those questions and one student's responses are provided in Figure 14.22.

Complete Self-Check 14.5: Mathematical Modeling

Algebraic Thinking across the Curriculum

One reason the phrase "algebraic thinking" is used instead of "algebra" is that the practice of looking for patterns, regularity, and generalizations goes beyond curriculum topics that are usually categorized as algebra topics. You have already experienced some of this integration—the strong connection between number and algebra (e.g., properties and generalizations), geometric growing patterns. Here we briefly share a few more.

Geometry, Measurement and Algebra

Soares, Blanton, and Kaput (2006) describe how to "algebraify" the elementary curriculum. One measurement example they give uses the children's book *Spaghetti and Meatballs for All* (https://www.youtube.com/watch?v=jN_GmgeU5cw) by Marilyn Burns looking at the increasing number of chairs needed given the growing number of tables.

Geometric formulas relate various dimensions, areas, and volumes of shapes. Each of these formulas involves at least one functional relationship. Consider any familiar formula for measuring a geometric shape. For example, the circumference of a circle is $c = 2\pi r$. The radius is the independent variable and circumference is the dependent variable. We can say that the circumference is dependent on the radius. Even nonlinear formulas like volume of a cone ($V = \frac{1}{3}\pi r^2 h$) are functions. Here the volume is a function of both the height of the cone and the radius. If the radius is held constant, the volume is a function of the height. Similarly, for a fixed height, the volume is a function of the radius.

The following activity explores how the volume of a box varies as a result of changing the dimensions.

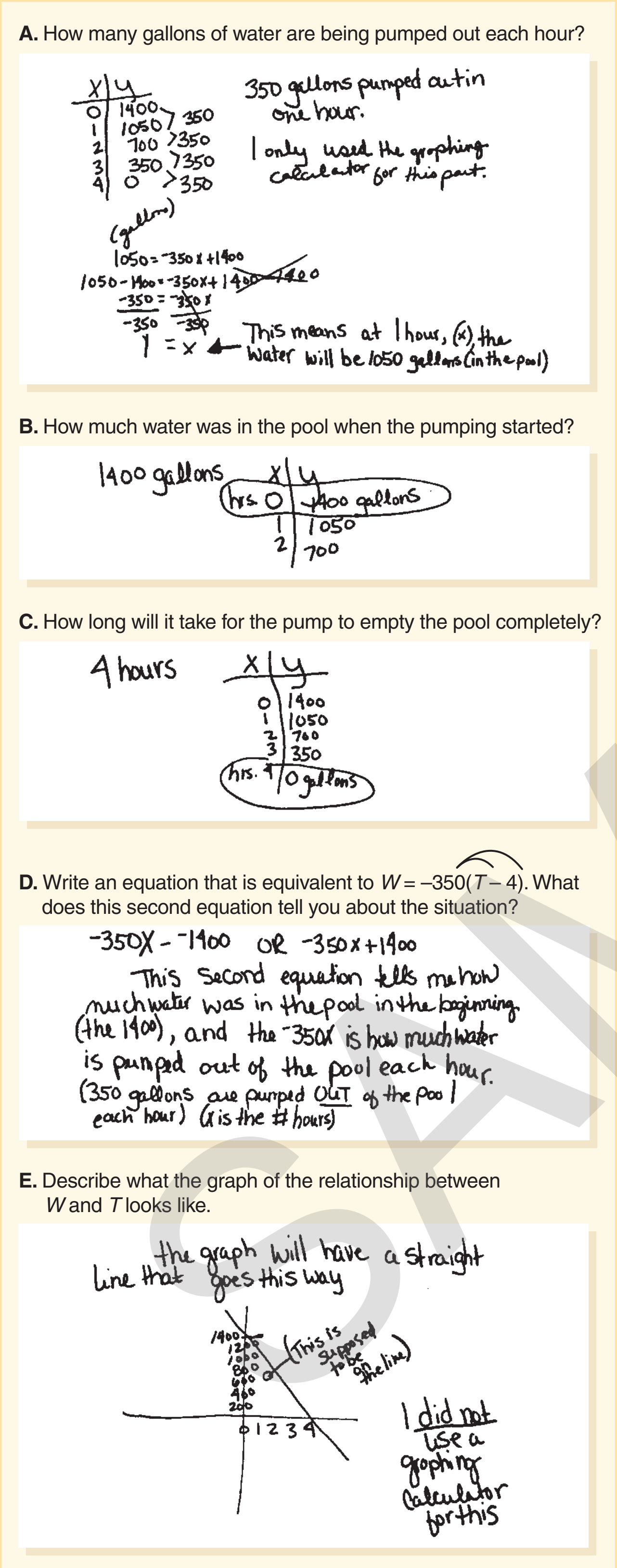
Activity 14.21

CCSS-M: 5.MD.C.5b; 6.EE.A.2c; 6.G.A.2

Designing the Largest Box

Give each student or pair of students a piece of card stock. Explain that they are to cut out a square from each corner using an exact measurement. All four squares must be the same size. Assign different lengths for the squares that are cut out (e.g., 2 cm, 2.5 cm, 3 cm, and so on). Explain to students that after they cut out their four squares, they will fold up the four resulting flaps, and tape them together to form an open box. Have students calculate the volume of their box. Then, have students trade boxes and determine the volume of other boxes. The volume of the box will vary depending on the size of the squares (see Figure 14.23). Ask students to record their data in a table.

After they have recorded the data for several boxes, challenge students to write a formula that gives the volume of the box as a function of the size of the cutout squares. Use the function to determine what size the squares should be to create the box with the largest volume. Alternatively, make origami boxes using squares with various side lengths and see what the relationship is between the side length and the volume of the open box. (See DeYoung, 2009, for instructions for making the box and more on this idea. Or look on the Internet.)


A. How many gallons of water are being pumped out each hour?

x	y
0	1400
1	1050
2	700
3	350
4	0

350 350 350 350

350 gallons pumped out in one hour.

I only used the graphing calculator for this part.

(gallons)

$1050 = -350x + 1400$

$1050 - 1400 = -350x + 1400 - 1400$

$-350 = -350x$

$\frac{-350}{-350} = \frac{-350x}{-350}$

$1 = x$ ← This means at 1 hour, (x), the water will be 1050 gallons (in the pool)

B. How much water was in the pool when the pumping started?

1400 gallons

x	y
hrs 0	1400 gallons
1	1050
2	700

C. How long will it take for the pump to empty the pool completely?

4 hours

x	y
0	1400
1	1050
2	700
3	350
hrs. 4	0 gallons

D. Write an equation that is equivalent to $W = -350(T - 4)$. What does this second equation tell you about the situation?

$-350X - {}^-1400$ OR $-350x + 1400$

This second equation tells me how much water was in the pool in the beginning (the 1400), and the ^-350x is how much water is pumped out of the pool each hour. (350 gallons are pumped OUT of the pool each hour) (x is the # hours)

E. Describe what the graph of the relationship between *W* and *T* looks like.

the graph will have a straight line that goes this way

1400 1200 1000 800 600 400 200

0 1 2 3 4

(This is supposed to be a line)

I did not use a graphing calculator for this

FIGURE 14.22 One student's explanations of questions regarding what a mathematical model means.

Source: Figure 3 from Herbel-Eisenmann, B. A., & Philips, E. D. (2005). "Using Student Work to Develop Teachers' Knowledge of Algebra." *Mathematics Teaching in the Middle School, 11*(2), p. 65. Reprinted with permission. Copyright © 2005 by the National Council of Teachers of Mathematics. All rights reserved.

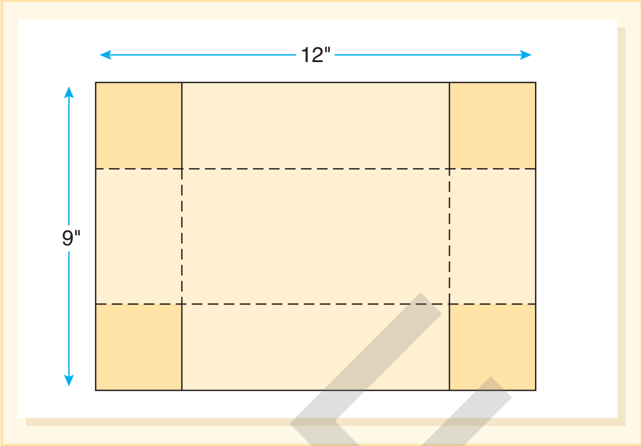


FIGURE 14.23 Cutting squares from cardstock. What size squares will result in an open box with the largest volume?

Data and Algebra. Data can be obtained from sports records, census reports, the business section of the newspaper, and many other sources. Students can gather data such as measurement examples or survey data. The Internet also has many sites where data can be found.

Experiments. There are many experiments that students can explore to see the functional relationships, if any, that exist between two variables. Gathering real data is an excellent way to engage a range of learners and to see how mathematics can be used to describe phenomena.

Data should be collected and then represented in a table or on a graph. The goal is to determine whether there is a relationship between the independent and dependent variables, and if so, whether it is linear or nonlinear, as in the following engaging experiments:

- How long would it take for 100 students standing in a row to complete a wave similar to those seen at football games? Experiment with different numbers of students from 5 to 25. Can the relationship predict how many students it would take for a given wave time?
- How far will a Matchbox car roll off of a ramp, based on the height the ramp is raised?
- How is the flight time of a paper airplane affected by the number of paper clips attached to the nose of the plane?
- What is the relationship between the number of dominoes in a row and the time required for them to fall over? (Use multiples of 100 dominoes.)
- Make wadded newspaper balls using different numbers of sheets of newspaper and a constant number of rubber bands to help hold the paper in a ball. What is the relationship between the number of sheets and the distance the ball can be thrown?
- What is the relationship between the number of drops of colored water dropped on a paper towel and the diameter of the spot? Is the relationship different for different brands of towels?

- How much weight can a toothpick bridge hold? Lay toothpicks in a bunch to span a 2-inch gap between two boards. From the toothpicks, hang a bag or other container into which weights can be added until the toothpicks break. Begin with only one toothpick (McCoy, 1997).

Experiments like these are fun and accessible to a wide range of learners. They also provide an opportunity for students to engage in experimental design—a perfect blend of mathematics and science.

Scatter Plots. Often in the real world, phenomena are observed that seem to suggest a functional relationship, but they are not necessarily as clean or as well defined as some of the situations we have described so far. In such cases, the data are generally plotted on a graph to produce a scatter plot of points. Two very good scatter plot generators can be found online at NLVM and NCES Kids' Zone.

A visual inspection of the scatter plot may suggest what kind of relationship, if any, exists. If a linear relationship seems to exist, for example, students can approximate a line of best fit or use graphing technology to do a linear regression to find the line of best fit (along with the equation).

Complete Self-Check 14.6: Algebraic Thinking across the Curriculum

REFLECTIONS ON CHAPTER 14

WRITING TO LEARN

1. What is functional thinking? How is it expressed?
2. What misconceptions or limited conceptions do students have regarding the equal sign? What causes these misconceptions, and how can instruction clear these up?
3. What misconceptions or limited conceptions do students have regarding variables? What causes these misconceptions, and how can instruction clear these up?
4. How does a graph of a pattern help develop algebraic thinking in students?
5. What is a recursive relationship? A correspondence relationship (explicit rule)? Where in a table for a growing pattern would you look for the recursive relationship?
6. What is mathematical modeling?

FOR DISCUSSION AND EXPLORATION

- The idea of having students make connections from arithmetic to algebra is the emphasis of algebra in the elementary grades. What examples can you find in the curriculum for taking an algorithm and presenting it in a way that it becomes a process for generalizing a rule?
- Try to understand the connections between algebraic thinking and functional thinking. How can you develop functional thinking in young students? What kind of models may be useful for this?

RESOURCES FOR CHAPTER 14

LITERATURE CONNECTIONS

The following three examples of books are excellent beginnings for patterns and chart building.

Anno's Magic Seeds *Anno (1994)*

Anno's Magic Seeds has several patterns. A wise man gives Jack two magic seeds, one to eat and one to plant. The planted seed will produce two new seeds by the following year. Several years later, Jack decides to plant both seeds. Then he has a family and starts to sell seeds. At each stage of the story, there is an opportunity to develop a chart and extend the current pattern into the future. Austin and Thompson (1997) describe how they used the story to develop patterns and charts with sixth- and seventh-grade students.

Bats on Parade *Appelt and Sweet (1999)*

This story includes the pattern of bats walking 1 by 1, then 2 by 2, and so on. One activity from this enjoyable book is determining the growing pattern of the number of bats given the array length (e.g., 3 for the 3×3 array). There is also one mouse, so this can be included in a second investigation. Activity Pages for these two ideas and two others can be found in Roy and Beckmann (2007).

Equal Shmequel *Kroll (2005)*

This story is about a mouse and her friends who want to play tug-of-war. To do so, they must determine how to make both sides equal so that the game is fair. In the end, they use a teeter-tooter to balance the weight of the friends. This focus on equal sides and balance make this a great book for focusing on the meaning of the equal sign.

Two of Everything: A Chinese Folktale *Hong (1993)*

(https://www.youtube.com/watch?v=TY_NP528ph4)

The magic pot discovered by Mr. Haktak doubles whatever goes in it, including his wife! This idea of input–output is great for exploring functions from grades 2 through 8; just vary the rule of the magic pot from doubling to something more complex. For more details and handouts, see Suh (2007a) and Wickett and colleagues (2002).

RECOMMENDED READINGS

Articles

Kalman, R. (2008). Teaching algebra without algebra. *Mathematics Teaching in the Middle School*, *13*(6), 334–339.

This article includes three contexts that involve simplifying equations and effectively explains how to make sense of the simplification by relating it to the context. An excellent resource for helping middle school students make sense of symbols and properties.

Leavy, A., Hourigan, M., & McMahon, A. (2013). Early understanding of equality. *Teaching Children Mathematics*, *20*(4), 247–252.

Nine strategies are shared for helping students strengthen their understanding of the equal sign. Each suggestion includes specific activity suggestions.

Molina, M., & Ambrose, R. C. (2006). Fostering relational thinking while negotiating the meaning of the equals sign. *Teaching Children Mathematics*, *13*(2), 111–117.

This article helps us understand the conceptual considerations related to the equal sign while simultaneously illustrating the value of errors and misconceptions in creating opportunities for learning.

Books

Essential Understandings Series (*Expressions, Equations, and Functions: Grades 6–8* (2011) and *Algebraic Thinking: Grades 3–5* (2011)). Reston, VA: NCTM.

Each of these books provides a teacher-friendly discussion of the big ideas of algebra. Interwoven are excellent tasks to use with students.

Blanton, M. L. (2008). *Algebra and the elementary classroom.* Portsmouth, NH: Heinemann.

This is an excellent book for teachers at all levels—full of rich problems to use and helpful for expanding the reader's understanding of algebra. Great for book study.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

This book is a detailed look at helping students in the primary grades develop the thinking and create the generalizations of algebra. The included CD shows classroom-based examples of the ideas discussed.

Fosnot, C. T., & Jacob, B. (2010). *Young mathematicians at work: Constructing algebra.* Portsmouth, NH: Heinemann.

Like the other books in the series, this is a gem. Full of classroom vignettes and examples that will enrich your understanding of how algebra can support arithmetic (and vice versa).

Greenes, C. E., & Rubenstein, R. (Eds.). (2008). *Algebra and algebraic thinking in school mathematics.* NCTM 70th Yearbook. Reston, VA: NCTM.

NCTM yearbooks are always excellent collections of articles for grades pre-K–12. This one is no exception, offering a wealth of thought-provoking and helpful articles about algebraic thinking.